✸*Part 1* Irish Mathematical Olympiad (IrishMO)

1st Irish 1988

1. One face of a pyramid with square base and all edges 2 is glued to a face of a regular tetrahedron with edge length 2 to form a polyhedron. What is the total edge length of the polyhedron?

2. P is a point on the circumcircle of the square ABCD between C and D. Show that $PA^2 - PB^2 = PB \cdot PD - PA \cdot PC$.

3. E is the midpoint of the arc BC of the circumcircle of the triangle ABC (on the opposite side of the line BC to A). DE is a diameter. Show that $\angle DEA$ is half the difference between the $\angle B$ and $\angle C$.

4. The triangle ABC (with sidelengths a, b, c as usual) satisfies $\log(a^2) = \log(b^2) + \log(c^2) - \log(2bc \cos A)$. What can we say about the triangle?

5. Let $X = \{1, 2, 3, 4, 5, 6, 7\}$. How many 7-tuples $(X_1, X_2, X_3, X_4, X_5, X_6, X_7)$ are there such that each X_i is a different subset of X with three elements and the union of the X_i is X?

6. Each member of the sequence $a_1, a_2, \ldots, a_n$ belongs to the set $\{1, 2, \ldots, n-1\}$ and $a_1 + a_2 + \ldots + a_n < 2n$. Show that we can find a subsequence with sum n.

7. Put $f(x) = x^3 - x$. Show that the set of positive real A such that for some real x we have $f(x + A) = f(x)$ is the interval $(0, 2]$.

8. The sequence of nonzero reals $x_1, x_2, x_3, \ldots$ satisfies $x_n = x_{n-2}x_{n-1}/(2x_{n-2} - x_{n-1})$ for all $n > 2$. For which (x_1, x_2) does the sequence contain infinitely many integral terms?

9. The year 1978 had the property that 19 + 78 = 97. In other words the sum of the number formed by the first two digits and the number formed by the last two digits equals the number formed by the middle two digits. Find the closest years either side of 1978 with the same property.

10. Show that $(1 + x)^n \geq (1 - x)^n + 2nx(1 - x^2)^{(n-1)/2}$ for all $0 \leq x \leq 1$ and all positive integers n.

11. Given a positive real k, for which real x_0 does the sequence $x_0, x_1, x_2, \ldots$ defined by

$x_{n+1} = x_n(2 - k x_n)$ converge to 1/k?

12. Show that for n a positive integer we have $\cos^4 k + \cos^4 2k + \dots \cos^4 nk = 3n/8 - 5/16$ where $k = \pi/(2n+1)$.

13. ABC is a triangle with AB = 2·AC and E is the midpoint of AB. The point F lies on the line EC and the point G lies on the line BC such that A, F, G are collinear and FG = AC. Show that $AG^3 = AB \cdot CE^2$.

14. $a_1, a_2, \dots, a_n$ are integers and m < n is a positive integer. Put $S_i = a_i + a_{i+1} + \dots + a_{i+m}$, and $T_i = a_{m+i} + a_{m+i+1} + \dots + a_{n-1+i}$, for where we use the usual cyclic subscript convention, whereby subscripts are reduced to the range 1, 2, ... , n by subtracting multiples of n as necessary. Let m(h, k) be the number of elements i in {1, 2, ... , n} for which $S_i = h$ mod 4 and $T_i = b$ mod 4. Show that m(1, 3) = m(3, 1) mod 4 iff m(2, 2) is even.

15. X is a finite set. $X_1, X_2, \dots, X_n$ are distinct subsets of X (n > 1), each with 11 elements, such that the intersection of any two subsets has just one element and given any two elements of X, there is an X_i containing them both. Find n.

2nd Irish 1989

A1. S is a square side 1. The points A, B, C, D lie on the sides of S in that order, and each side of S contains at least one of A, B, C, D. Show that $2 \le AB^2 + BC^2 + CD^2 + DA^2 \le 4$.

A2. A *sumsquare* is a 3 x 3 array of positive integers such that each row, each column and each of the two main diagonals has sum m. Show that m must be a multiple of 3 and that the largest entry in the array is at most 2m/3 - 1.

A3. The sequence $a_1, a_2, a_3, \dots$ is defined by $a_1 = 1$, $a_{2n} = a_n$, $a_{2n+1} = a_{2n} + 1$. Find the largest value in $a_1, a_2, \dots, a_{1989}$ and the number of times it occurs.

A4. n^2 ends with m equal non-zero digits (in base 10). What is the largest possible value of m?

A5. An n-digit number has the property that if you cyclically permute its digits it is always divisible by 1989. What is the smallest possible value of n? What is the smallest such number? [If we cyclically permute the digits of 3701 we get 7013, 137, 1370, and 3701.]

B1. L is a fixed line, A a fixed point, $k > 0$ a fixed real. P is a variable point on L, Q is the point on the ray AP such that $AP \cdot AQ = k^2$. Find the locus of Q.

B2. Each of n people has a unique piece of information. They wish to share the information. A person may pass another person a message containing all the pieces of information that he has. What is the smallest number of messages that must be passed so that each person ends up with all n pieces of information? For example, if A, B, C start by knowing a, b, c respectively. Then four messages suffice: A passes a to B; B passes a and b to C; C passes a, b and c to A; C passes a, b, c to B.

B3. Let k be the product of the distances from P to the sides of the triangle ABC. Show that if P is inside ABC, then $AB \cdot BC \cdot CA \geq 8k$ with equality iff ABC is equilateral.

B4. Show that $(n + \sqrt{(n^2 + 1)})^{1/3} + (n - \sqrt{(n^2 + 1)})^{1/3}$ is a positive integer iff $n = m(m^2 + 3)/2$ for some positive integer m.

B5.(a) Show that $2nCn < 2^{2n}$ and that it is divisible by all primes p such that $n < p < 2n$ (where $2nCn = (2n)! /(n!\, n!)$).

(b) Let $\pi(x)$ denote the number of primes $\leq x$. Show that for $n > 2$ we have $\pi(2n) < \pi(n) + 2n/\log_2 n$ and $\pi(2^n) < (1/n)\, 2^{n+1} \log_2(n-1)$. Deduce that for $x \geq 8$, $\pi(x) < (4x/\log_2 x) \log_2(\log_2 x)$.

3rd Irish 1990

1. Find the number of rectangles with sides parallel to the axes whose vertices are all of the form (a, b) with a and b integers such that $0 \leq a, b \leq n$.

2. The sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 2$, a_n is the largest prime divisor of $a_1a_2 \ldots a_{n-1} + 1$. Show that 5 does not occur in the sequence.

3. Does there exist a function f(n) on the positive integers which takes positive integer values and satisfies $f(n) = f(\, f(n-1)\,) + f(\, f(n+1)\,)$ for all $n > 1$?

4. Find the largest n for which we can find a real number x satisfying:
$2^1 < x^1 + x^2 < 2^2$
$2^2 < x^2 + x^3 < 2^3$
...
$2^n < x^n + x^{n+1} < 2^{n+1}$.

5. In the triangle ABC, $\angle A = 90^o$. X is the foot of the perpendicular from A, and D is the reflection of A in B. Y is the midpoint of XC. Show that DX is perpendicular to AY.

6. If all $a_n = \pm 1$ and $a_1a_2 + a_2a_3 + ... a_{n-1}a_n + a_na_1 = 0$, show that n is a multiple of 4.

7. Show that $1/3^3 + 1/4^3 + ... + 1/n^3 < 1/12$.

8. $p_1 < p_2 < ... < p_{15}$ are primes forming an arithmetic progression, show that the difference must be a multiple of $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$.

9. Let $a_n = 2 \cos(t/2^n) - 1$. Simplify $a_1a_2 ... a_n$ and deduce that it tends to $(2 \cos t + 1)/3$.

10. Let T be the set of all (2k-1)-tuples whose entries are all 0 or 1. There is a subset S of T with 2^k elements such that given any element x of T, there is a unique element of S which disagrees with x in at most 3 positions. If k > 5, show that it must be 12.

4th Irish 1991

A1. Given three points X, Y, Z, show how to construct a triangle ABC which has circumcenter X, Y the midpoint of BC and BZ an altitude.

A2. Find all polynomials p(x) of degree ≤ n which satisfy $p(x^2) = p(x)^2$ for all real x.

A3. For any positive integer n, define f(n) = 10n, g(n) = 10n+4, and for any even positive integer n, define h(n) = n/2. Show that starting from 4 we can reach any positive integer by some finite sequence of the operations f, g, h.

A4. 8 people decide to hold daily meetings subject to the following rules. At least one person must attend each day. A different set of people must attend on different days. On day N for each $1 \le k < N$, at least one person must attend who was present on day k. How many days can the meetings be held?

A5. Find all polynomials $x^n + a_1x^{n-1} + ... + a_n$ such that each of the coefficients $a_1, a_2, ... , a_n$ is ±1 and all the roots are real.

B1. Prove that the sum of m consecutive squares cannot be a square for m = 3, 4, 5, 6. Give an example of 11 consecutive squares whose sum is a square.

B2. Define $a_n = (n^2 + 1)/\sqrt{(n^4 + 4)}$ for n = 1, 2, 3, ... , and let $b_n = a_1a_2 \ldots a_n$. Show that $b_n = (\sqrt{2}\ \sqrt{(n^2+1)})/\sqrt{(n^2+2n+2)}$, and hence that $1/(n+1)^3 < b_n/\sqrt{2} - n/(n+1) < 1/n^3$.

B3. ABC is a triangle and L is the line through C parallel to AB. The angle bisector of A meets BC at D and L at E. The angle bisector of B meets AC at F and L at G. If DE = FG show that CA = CB.

B4. P is the set of positive rationals. Find all functions f:P→P such that f(x) + f(1/x) = 1 and f(2x) = 2 f(f(x)) for all x.

B5. A non-empty subset S of the rationals satisfies: (1) $0 \notin S$; (2) if a, b $\in$ S, then a/b $\in$ S; (3) there is a non-zero rational q not in S such that if s is a non-zero rational not in S, then s = qt for some t $\in$ S. Show that every element of S is a sum of two elements of S.

5th Irish 1992

A1. Give a geometric description for the set of points (x, y) such that $t^2 + yt + x \geq 0$ for all real t satisfying $|t| \leq 1$.

A2. How many (x, y, z) satisfy $x^2 + y^2 + z^2 = 9$, $x^4 + y^4 + z^4 = 33$, $xyz = -4$?

A3. A has n elements. How many (B, C) are such that $\emptyset \neq B \subseteq C \subseteq A$?

A4. ABC is a triangle with circumradius R. A', B', C' are points on BC, CA, AB such that AA', BB', CC' are concurrent. Show that the AB'·BC'·CA'/area A'B'C' = 2R.

A5. A triangle has two vertices with rational coordinates. Show that the third vertex has rational coordinates iff each angle X of the triangle has $X = 90^o$ or tan X rational.

B1. Let $m = \sum k^3$, where the sum is taken over $1 \leq k < n$ such that k is relatively prime to n. Show that m is a multiple of n.

B2. The digital root of a positive integer is obtained by repeatedly taking the product of the digits until we get a single-digit number. For example 24378 → 1344 → 48 → 32 → 6. Show that if n has digital root 1, then all its digits are 1.

B3. All three roots of $az^3 + bz^2 + cz + d$ have negative real part. Show that ab > 0, bc > ad > 0.

B4. Each diagonal of a convex pentagon divides the pentagon into a quadrilateral and a triangle of unit area. Find the area of the pentagon.

B5. Show that for any positive reals a_i, b_i, we have $(a_1a_2 \dots a_n)^{1/n} + (b_1b_2 \dots b_n)^{1/n} \le ((a_1+b_1)(a_2+b_2) \dots (a_n+b_n))^{1/n}$ with equality iff $a_1/b_1 = a_2/b_2 = \dots = a_n/b_n$.

6th Irish 1993

A1. The real numbers x, y satisfy $x^3 - 3x^2 + 5x - 17 = 0$, $y^3 - 3y^2 + 5y + 11 = 0$. Find x + y.

A2. Find which positive integers can be written as the sum and product of the same sequence of two or more positive integers. (For example $10 = 5+2+1+1+1 = 5\cdot2\cdot1\cdot1\cdot1$).

A3. A triangle ABC has fixed incircle. BC touches the incircle at the fixed point P. B and C are varied so that PB·PC is constant. Find the locus of A.

A4. The polynomial $x^n + a_{n-1}x^{n-1} + \dots + a_0$ has real coefficients. All its roots are real and lie in the interval (0, 1). Also $f(1) = |f(0)|$. Show that the product of the roots does not exceed $1/2^n$.

A5. The points z_1, z_2, z_3, z_4, z_5 form a convex pentagon P in the complex plane. The origin and the points $\alpha z_1, \dots, \alpha z_5$ all lie inside the pentagon. Show that $|\alpha| \le 1$ and $\mathrm{Re}(\alpha) + \mathrm{Im}(\alpha)\tan(\pi/5) \le 1$.

B1. Given 5 lattice points in the plane, show that at least one pair of points has a distinct lattice point on the segment joining them.

B2. $a_1, a_2, \dots, a_n$ are distinct reals. $b_1, b_2, \dots, b_n$ are reals. There is a real number α such that $\prod_{1\le k\le n} (a_i + b_k) = \alpha$ for i = 1, 2, ... , n. Show that there is a real β such that $\prod_{1\le k\le n} (a_k + b_j) = \beta$ for j = 1, 2, ... , n.

B3. Given positive integers $r \le n$, show that $\sum_d$ (n-r+1)Cd (r-1)C(d-1) = nCr, where nCr denotes the usual binomial coefficient and the sum is taken over all positive $d \le n-r+1$ and $\le r$.

B4. Show that $\sin x + (\sin 3x)/3 + (\sin 5x)/5 + \dots + (\sin(2n-1)x)/(2n-1) > 0$ for all x in $(0, \pi)$.

B5. An m x n rectangle is divided into unit squares. Show that a diagonal of the rectangle intersects m + n - gcd(m,n) of the squares. An a x b x c box is divided into unit cubes. How many cubes does a long diagonal of the box intersect?

7th Irish 1994

A1. m, n are positive integers with n > 3 and $m^2 + n^4 = 2(m-6)^2 + 2(n+1)^2$. Prove that $m^2 + n^4 = 1994$.

A2. B is an arbitrary point on the segment AC. Equilateral triangles are drawn as shown. Show that their centers form an equilateral triangle whose center lies on AC.

A3. Find all real polynomials p(x) satisfying $p(x^2) = p(x)p(x-1)$ for all x.

A4. An n x n array of integers has each entry 0 or 1. Find the number of arrays with an even number of 1s in every row and column.

A5. The sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 2$, $a_{n+1} = a_n^2 - a_n + 1$. Show that $1/a_1 + 1/a_2 + \ldots + 1/a_n$ lies in the interval $(1-½^N, 1-½^{2N})$, where $N = 2^{n-1}$.

B1. The sequence $x_1, x_2, x_3, \ldots$ is defined by $x_1 = 2$, $nx_n = 2(2n-1)x_{n-1}$. Show that every term is integral.

B2. p, q, r are distinct reals such that q = p(4-p), r = q(4-q), p = r(4-r). Find all possible values of p+q+r.

B3. Prove that $n((n+1)^{2/n} - 1) < \sum_1^n (2i+1)/i^2 < n(1 - 1/n^{2/(n-1)}) + 4$.

B4. w, a, b, c are distinct real numbers such that the equations:
$x + y + z = 1$
$xa^2 + yb^2 + zc^2 = w^2$
$xa^3 + yb^3 + zc^3 = w^3$
$xa^4 + yb^4 + zc^4 = w^4$
have a real solution x, y, z. Express w in terms of a, b, c.

B5. A square is partitioned into n convex polygons. Find the maximum number of edges in the resulting figure. You may assume Euler's formula for a polyhedron: V + F = E + 2, where V is the no. of vertices, F the no. of faces and E the no. of edges.

8th Irish 1995

A1. There are n^2 students in a class. Each week they are arranged into n teams of n players. No two students can be in the same team in more than one week. Show that the arrangement can last for at most n+1 weeks.

A2. Find all integers n for which $x^2 + nxy + y^2 = 1$ has infinitely many distinct integer solutions x, y.

A3. X lies on the line segment AD. B is a point in the plane such that $\angle ABX > 120^o$. C is a point on the line segment BX, show that $(AB + BC + CD) \leq 2AD/\sqrt{3}$.

A4. X_k is the point (k, 0). There are initially 2n+1 disks, all at X_0. A move is to take two disks from X_k and to move one to X_{k-1} and the other to X_{k+1}. Show that whatever moves are chosen, after n(n+1)(2n+1)/6 moves there is one disk at X_k for $|k| \leq n$.

A5. Find all real-valued functions f(x) such that xf(x) - yf(y) = (x-y) f(x+y) for all real x, y.

B1. Show that for every positive integer n, $n^n \leq (n!)^2 \leq ((n+1)(n+2)/6)^n$.

B2. a, b, c are complex numbers. All roots of $z^3 + az^2 + bz + c = 0$ satisfy $|z| = 1$. Show that all roots of $z^3 + |a|z^2 + |b|z + |c| = 0$ also satisfy $|z| = 1$.

B3. S is the square $\{(x,y) : 0 \leq x, y \leq 1\}$. For each $0 < t < 1$, C_t is the set of points (x, y) in S such that $x/t + y/(1-t) \geq 1$. Show that the set $\cap C_t$ is the points (x, y) in S such that $\sqrt{x} + \sqrt{y} \geq 1$.

B4. Given points P, Q, R show how to construct a triangle ABC such that P, Q, R are on BC, CA, AB respectively and P is the midpoint of BC, CQ/QA = AR/RB = 2. You may assume that P, Q, R are positioned so that such a triangle exists.

B5. $n < 1995$ and n = abcd, where a, b, c, d are distinct primes. The positive divisors of n are $1 = d_1 < d_2 < ... < d_{16} = n$. Show that $d_9 - d_8 \neq 22$.

9th Irish 1996

A1. Find gcd(n!+1, (n+1)!).

A2. Let s(n) denote the sum of the digits of n. Show that $s(2n) \le 2s(n) \le 10s(2n)$ and that there is a k such that $s(k) = 1996\ s(3k)$.

A3. R denotes the reals. $f : [0,1] \to R$ satisfies $f(1) = 1$, $f(x) \ge 0$ for all $x \in [0,1]$, and if x, y, x+y all $\in [0,1]$, then $f(x+y) \ge f(x) + f(y)$. Show that $f(x) \le 2x$ for all $x \in [0,1]$.

A4. ABC is any triangle. D, E are constructed as shown so that ABD and ACE are right-angled isosceles triangles, and F is the midpoint of BC. Show that DEF is a right-angled isosceles triangle.

A5. Show how to dissect a square into at most 5 pieces so that the pieces can be reassembled to form three squares all of different size.

B1. The Fibonacci sequence $F_0, F_1, F_2, \ldots$ is defined by $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$. Show that $F_{n+60} - F_n$ is divisible by 10 for all n, but for any $1 \le k < 60$ there is some n such that $F_{n+k} - F_n$ is not divisible by 10. Similarly, show that $F_{n+300} - F_n$ is divisible by 100 for all n, but for any $1 \le k < 300$ there is some n such that $F_{n+k} - F_n$ is not divisible by 100.

B2. Show that $2^{1/2}4^{1/4}8^{1/8} \ldots (2^n)^{1/2n} < 4$.

B3. If p is a prime, show that $2^p + 3^p$ cannot be an nth power (for $n > 1$).

B4. ABC is an acute-angled triangle. The altitudes are AD, BE, CF. The feet of the perpendiculars from A, B, C to EF, FD, DE respectively are P, Q, R. Show that AP, BQ, CR are concurrent.

B5. 33 disks are placed on a 5 x 9 board, at most one disk per square. At each step every disk is moved once so that after the step there is at most one disk per square. Each disk is moved alternately one square up/down and one square left/right. So a particular disk might be moved L,U,L,D,L,D,R,U ... in successive steps. Prove that only finitely many steps are possible. Show that with 32 disks it is possible to have infinitely many steps.

10th Irish 1997

A1. Find all integer solutions to $1 + 1996m + 1998n = mn$.

A2. ABC is an equilateral triangle. M is a point inside the triangle. D, E, F are the feet of the perpendiculars from M to BC, CA, AB. Find the locus of M such that $\angle FDE = 90^o$.

A3. Find all polynomials p(x) such that (x-16) p(2x) = (16x-16) p(x).

A4. a, b, c are non-negative reals such that $a + b + c \geq abc$. Show that $a^2 + b^2 + c^2 \geq abc$.

A5. Let S denote the set of odd integers > 1. For $x \in S$, define f(x) to be the largest integer such that $2^{f(x)} < x$. For a, b $\in$ S define $a * b = a + 2^{f(a)-1}(b-3)$. For example, f(5) = 2, so 5 * 7 = 5 + 2(7-3) = 13. Similarly, f(7) = 2, so 7 * 5 = 7 + 2(5-3) = 11. Show that a * b is always an odd integer > 1 and that the operation * is associative.

B1. Let $\sigma(n)$ denote the sum of the positive divisors of n. Show that if $\sigma(n) > 2n$, then $\sigma(mn) > 2mn$ for any m.

B2. The quadrilateral ABCD has an inscribed circle. $\angle A = \angle B = 120^o$, $\angle C = 30^o$ and BC = 1. Find AD.

B3. A subset of {0, 1, 2, ... , 1997} has more than 1000 elements. Show that it must contain a power of 2 or two distinct elements whose sum is a power of 2.

B4. How many 1000-digit positive integers have all digits odd, and are such that any two adjacent digits differ by 2?

B5. p is an odd prime. We say n satisfies K_p if the set {1, 2, ... , n} can be partitioned into p disjoint parts, such that the sum of the elements in each part is the same. For example, 5 satisfies K_3 because {1, 2, 3, 4, 5} = {1, 4} &cup {2, 3} $\cup$ {5}. Show that if n satisfies K_p, then n or n+1 is a multiple of p. Show that if n is a multiple of 2p, then n satisfies K_p.

11th Irish 1998

A1. Show that $x^8 - x^5 - 1/x + 1/x^4 \geq 0$ for all $x \neq 0$.

A2. P is a point inside an equilateral triangle. Its distances from the vertices are 3, 4, 5. Find the area of the triangle.

A3. Show that the 4 digit number mnmn cannot be a cube in base 10. Find the smallest base b > 1 for which it can be a cube.

A4. Show that 7 disks radius 1 can be arranged to cover a disk radius 2.

A5. x is real and $x^n - x$ is an integer for $n = 2$ and some $n > 2$. Show that x must be an integer.

B1. Find all positive integers n with exactly 16 positive divisors $1 = d_1 < d_2 < ... < d_{16} = n$ such that $d_6 = 18$ and $d_9 - d_8 = 17$.

B2. Show that for positive reals a, b, c we have $9/(a+b+c) \le 2/(a+b) + 2/(b+c) + 2/(c+a) \le 1/a + 1/b + 1/c$.

B3. Let N be the set of positive integers. Show that we can partition N into three disjoint parts such that if $|m-n| = 2$ or 5, then m and n are in different parts. Show that we can partition N into four disjoint parts such that if $|m-n| = 2$, 3 or 5, then m and n are in different parts, but that this is not possible with only three disjoint parts.

B4. The sequence $x_0, x_1, x_2, ...$ is defined by $x_0 = a$, $x_1 = b$, $x_{n+2} = (1+x_{n+1})/x_n$. Find x_{1998}.

B5. Find the smallest possible perimeter for a triangle ABC with integer sides such that $\angle A = 2\angle B$ and $\angle C > 90^\circ$.

12th Irish 1999

A1. Find all real solutions to $x^2/(x+1-\sqrt{(x+1)})^2 < (x^2+3x+18)/(x+1)^2$.

A2. The Fibonacci sequence is defined by $F_0 = 1$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$. Show that some Fibonacci number is divisible by 1000.

A3. In the triangle ABC, AD is an altitude, BE is an angle bisector and CF is a median. Show that they are concurrent iff $a^2(a-c) = (b^2-c^2)(a+c)$.

A4. Show that a 10000 x 10000 board can be tiled by 1 x 3 tiles and a 2 x 2 tile placed centrally, but not if the 2 x 2 tile is placed in a corner.

A5. The sequence $u_0, u_1, u_2, ...$ is defined as follows. $u_0 = 0$, $u_1 = 1$, and u_{n+1} is the smallest integer $> u_n$ such that there is no arithmetic progression u_i, u_j, u_{n+1} with $i < j < n+1$. Find u_{100}.

B1. Solve: $y^2 = (x+8)(x^2+2)$ and $y^2 - (8+4x)y + (16+16x-5x^2) = 0$.

B2. f(n) is a function defined on the positive integers with positive integer values such that $f(ab) = f(a)f(b)$ when a, b are relatively prime and $f(p+q) = f(p)+f(q)$ for all primes p, q. Show that $f(2) = 2$, $f(3) = 3$ and $f(1999) = 1999$.

B3. Show that $a^2/(a+b) + b^2/(b+c) + c^2/(c+d) + d^2/(d+a) \geq 1/2$ for positive reals a, b, c, d such that $a + b + c + d = 1$, and that we have equality iff $a = b = c = d$.

B4. Let d(n) be the number of positive divisors of n. Find all n such that $n = d(n)^4$.

B5. ABCDEF is a convex hexagon such that AB = BC, CD = DE, EF = FA and $\angle B + \angle D + \angle F = 360^o$. Show that the perpendiculars from A to FB, C to BD, and E to DF are concurrent.

13th Irish 2000

A1. Let S be the set of all numbers of the form n^2+n+1. Show that the product of n^2+n+1 and $(n+1)^2+(n+1)+1$ is in S, but give an example of a, b $\in$ S with ab $\notin$ S.

A2. ABCDE is a regular pentagon side 1. F is the midpoint of AB. G, H are points on DC, DE respectively such that $\angle DFG = \angle DFH = 30^o$. Show that FGH is equilateral and $GH = 2 \cos 18^o \cos^2 36^o/\cos 6^o$. A square is inscribed in FGH with one side on GH. Show that its side has length $GH\sqrt{3}/(2+\sqrt{3})$.

A3. Let $f(n) = 5n^{13} + 13n^5 + 9an$. Find the smallest positive integer a such that f(n) is divisible by 65 for every integer n.

A4. A strictly increasing sequence $a_1 < a_2 < ... < a_M$ is called a *weak AP* if we can find an arithmetic progression $x_0, x_1, ... , x_M$ such that $x_{n-1} \leq a_n < x_n$ for n = 1, 2, ... , M. Show that any strictly increasing sequence of length 3 is a weak AP. Show that any subset of {0, 1, 2, ... , 999} with 730 members has a weak AP of length 10.

A5. Let $y = x^2 + 2px + q$ be a parabola which meets the x- and y-axes in three distinct points. Let C_{pq} be the circle through these points. Show that all circles C_{pq} pass through a common point.

B1. Show that $x^2y^2(x^2+y^2) \leq 2$ for positive reals x, y such that x+y = 2.

B2. ABCD is a cyclic quadrilateral with circumradius R, side lengths a, b, c, d and area S. Show that $16R^2S^2 = (ab+cd)(ac+bd)(ad+bc)$. Deduce that $RS\sqrt{2} \geq (abcd)^{3/4}$ with equality iff ABCD is a square.

B3. For each positive integer n, find all positive integers m which can be written as $1/a_1 + 2/a_2 + ... + n/a_n$ for some positive integers $a_1 < a_2 < ... < a_n$.

B4. Show that in any set of 10 consecutive integers there is one which is relatively prime to each of the others.

B5. p(x) is a plynomial with non-negative real coefficients such that p(4) = 2, p(16) = 8. Show that $p(8) \leq 4$ and find all polynomials where equality holds.

14th Irish 2001

A1. Find all solutions to $a! + b! + c! = 2^n$.

A2. ABC is a triangle. Show that the medians BD and CE are perpendicular iff $b^2 + c^2 = 5a^2$.

A3. p is an odd prime which can be written as a difference of two fifth powers. Show that $\sqrt{((4p+1)/5)} = (n^2+1)/2$ for some odd integer n.

A4. Show that $2n/(3n+1) \leq \sum_{n<k\leq 2n} 1/k \leq (3n+1)/(4n+4)$.

A5. Show that $(a^2b^2(a+b)^2/4)^{1/3} \leq (a^2+10ab+b^2)/12$ for all reals a, b such that ab > 0. When do we have equality? Find all real numbers a, b for which $(a^2b^2(a+b)^2/4)^{1/3} \leq (a^2+ab+b^2)/3$.

B1. Find the smallest positive integer m for which $55^n + m32^n$ is a multiple of 2001 for some odd n.

B2. Three circles each have 10 black beads and 10 white beads randomly arranged on them. Show that we can always rotate the beads around the circles so that in 5 corresponding positions the beads have the same color.

B3. P is a point on the altitude AD of the triangle ABC. The lines BP, CP meet CA, AB at E, F respectively. Show that AD bisects $\angle EDF$.

B4. Find all non-negative reals for which $(13 + \sqrt{x})^{1/3} + (13 - \sqrt{x})^{1/3}$ is an integer.

B5. Let N be the set of positive integers. Find all functions $f : N \to N$ such that $f(m + f(n)) = f(m) + n$ for all m, n.

15th Irish 2002

A1. The triangle ABC has a, b, c = 29, 21, 20 respectively. The points D, E lie on the segment BC with BD = 8, DE = 12, EC = 9. Find $\angle DAE$.

A2. A graph has n points. Each point has degree at most 3. If there is no edge between two points, then there is a third point joined to them both. What is the maximum possible value of n? What is the maximum if the graph contains a triangle?

A3. Find all positive integer solutions to $p(p+3) + q(q+3) = n(n+3)$, where p and q are primes.

A4. Define the sequence $a_1, a_2, a_3, \ldots$ by $a_1 = a_2 = a_3 = 1$, $a_{n+3} = (a_{n+2}a_{n+1} + 2)/a_n$. Show that all terms are integers.

A5. Show that $x/(1-x) + y/(1-y) + z/(1-z) \geq 3(xyz)^{1/3}/(1 - (xyz)^{1/3})$ for positive reals x, y, z all < 1.

B1. For which n can we find a cyclic shift $a_1, a_2, \ldots, a_n$ of 1, 2, 3, ... , n (ie i, i+1, i+2, ... , n, 1, 2, ... , i-1 for some i) and a permutation $b_1, b_2, \ldots, b_n$ of 1, 2, 3, ... , n such that $1 + a_1 + b_1 = 2 + a_2 + b_2 = \ldots = n + a_n + b_n$?

B2. $n = p \cdot q \cdot r \cdot s$, where p, q, r, s are distinct primes such that $s = p + r$, $p(p + q + r + s) = r(s - q)$ and $qs = 1 + qr + s$. Find n.

B3. Let Q be the rationals. Find all functions $f : Q \to Q$ such that $f(x + f(y)) = f(x) + y$ for all x, y.

B4. Show that $k^n - [k^n] = 1 - 1/k^n$, where $k = 2 + \sqrt{3}$.

B5. The incircle of the triangle ABC touches BC at D and AC at E. The sides have integral lengths and $|AD^2 - BE^2| \leq 2$. Show that AC = BC.

16th Irish 2003

A1. Find all integral solutions to $(m^2 + n)(m + n^2) = (m + n)^3$.

A2. QB is a chord of the circle parallel to the diameter PA. The lines PB and QA meet at R. S is taken so that PORS is a parallelogram (where O is the center of the circle). Show that SP = SQ.

A3. Find $([1^{1/2}] - [1_{1/3}]) + ([2_{1/2}] - [2_{1/3}]) + ... + ([2003^{1/2} - [2003^{1/3}])$.

A4. A, B, C, D, E, F, G, H compete in a chess tournament. Each pair plays at most once and no five players all play each other. Write a possible arrangment of 24 games which satisfies the conditions and show that no arrangement of 25 games works.

A5. Let R be the reals and R^+ the positive reals. Show that there is no function $f : R^+ \to R$ such that $f(y) > (y - x) f(x)^2$ for all x, y such that y > x.

B1. A triangle has side lengths a, b, c with sum 2. Show that $1 \le ab + bc + ca - abc \le 1 + 1/27$.

B2. ABCD is a quadrilateral. The feet of the perpendiculars from D to AB, BC are P, Q respectively, and the feet of the perpendiculars from B to AD, CD are R, S respectively. Show that if $\angle PSR = \angle SPQ$, then PR = QS.

B3. Find all integer solutions to $m^2 + 2m = n^4 + 20n^3 + 104n^2 + 40n + 2003$.

B4. Given real positive a, b, find the largest real c such that $c \le \max(ax + 1/(ax), bx + 1/bx)$ for all positive real x.

B5. N distinct integers are to be chosen from {1, 2, ... , 2003} so that no two of the chosen integers differ by 10. How many ways can this be done for N = 1003? Show that it can be done in $(3 \cdot 5151 + 7 \cdot 1700)\ 101^7$ ways for N = 1002.

✸ *Part 2* Brasil Mathematical Olympiad(BrMO)

1st Brasil 1979

1. Show that if $a < b$ are in the interval $[0, \pi/2]$ then $a - \sin a < b - \sin b$. Is this true for $a < b$ in the interval $[\pi, 3\pi/2]$?

2. The remainder on dividing the polynomial $p(x)$ by $x^2 - (a+b)x + ab$ (where a and b are unequal) is $mx + n$. Find the coefficients m, n in terms of a, b. Find m, n for the case $p(x) = x^{200}$ divided by $x^2 - x - 2$ and show that they are integral.

3. The vertex C of the triangle ABC is allowed to vary along a line parallel to AB. Find the locus of the orthocenter.

4. Show that the number of positive integer solutions to $x_1 + 2^3x_2 + 3^3x_3 + ... + 10^3x_{10} = 3025$ (*) equals the number of non-negative integer solutions to the equation $y_1 + 2^3y_2 + 3^3y_3 + ... + 10^3y_{10} = 0$. Hence show that (*) has a unique solution in positive integers and find it.

5.(i) ABCD is a square with side 1. M is the midpoint of AB, and N is the midpoint of BC. The lines CM and DN meet at I. Find the area of the triangle CIN.

(ii) The midpoints of the sides AB, BC, CD, DA of the parallelogram ABCD are M, N, P, Q respectively. Each midpoint is joined to the two vertices not on its side. Show that the area outside the resulting 8-pointed star is 2/5 the area of the parallelogram.

(iii) ABC is a triangle with CA = CB and centroid G. Show that the area of AGB is 1/3 of the area of ABC.

(iv) Is (ii) true for all convex quadrilaterals ABCD?

2nd Brasil 1980

1. Box A contains black balls and box B contains white balls. Take a certain number of balls from A and place them in B. Then take the same number of balls from B and place them in A. Is the number of white balls in A then greater, equal to, or less than the number of black balls in B?

2. Show that for any positive integer $n > 2$ we can find n distinct positive integers such that the sum of their reciprocals is 1.

3. Given a triangle ABC and a point P_0 on the side AB. Construct points P_i, Q_i, R_i as follows. Q_i is the foot of the perpendicular from P_i to BC, R_i is the foot of the perpendicular from Q_i to AC and P_i is the foot of the perpendicular from R_{i-1} to AB.

Show that the points P_i converge to a point P on AB and show how to construct P.

4. Given 5 points of a sphere radius r, show that two of the points are a distance $\leq r\sqrt{2}$ apart.

3rd Brasil 1981

1. For which k does the system $x^2 - y^2 = 0$, $(x-k)^2 + y^2 = 1$ have exactly (1) two, (2) three real solutions?

2. Show that there are at least 3 and at most 4 powers of 2 with m digits. For which m are there 4?

3. Given a sheet of paper and the use of a rule, compass and pencil, show how to draw a straight line that passes through two given points, if the length of the ruler and the maximum opening of the compass are both less than half the distance between the two points. You may not fold the paper.

4. A graph has 100 points. Given any four points, there is one joined to the other three. Show that one point must be joined to all 99 other points. What is the smallest number possible of such points (that are joined to all the others)?

5. Two thieves stole a container of 8 liters of wine. How can they divide it into two parts of 4 liters each if all they have is a 3 liter container and a 5 liter container? Consider the general case of dividing m+n liters into two equal amounts, given a container of m liters and a container of n liters (where m and n are positive integers). Show that it is possible iff m+n is even and (m+n)/2 is divisible by gcd(m,n).

6. The centers of the faces of a cube form a regular octahedron of volume V. Through each vertex of the cube we may take the plane perpendicular to the long diagonal from the vertex. These planes also form a regular octahedron. Show that its volume is 27V.

4th Brasil 1982

1. The angles of the triangle ABC satisfy $\angle A/\angle C = \angle B/\angle A = 2$. The incenter is O. K, L are the excenters of the excircles opposite B and A respectively. Show that triangles ABC and OKL are similar.

2. Any positive integer n can be written in the form $n = 2^b(2c+1)$. We call 2c+1 the *odd part* of n. Given an odd integer n > 0, define the sequence $a_0, a_1, a_2, \ldots$ as follows: $a_0 = 2^n-1$, a_{k+1} is the odd part of $3a_k+1$. Find a_n.

3. S is a (k+1) x (k+1) array of lattice points. How many squares have their vertices in S?

4. Three numbered tiles are arranged in a tray as shown:
Show that we cannot interchange the 1 and the 3 by a sequence of moves where we slide a tile to the adjacent vacant space.

5. Show how to construct a line segment length $(a^4 + b^4)^{1/4}$ given segments length a and b.

6. Five spheres of radius r are inside a right circular cone. Four of the spheres lie on the base of the cone. Each touches two of the others and the sloping sides of the cone. The fifth sphere touches each of the other four and also the sloping sides of the cone. Find the volume of the cone.

5th Brasil 1983

1. Show that there are only finitely many solutions to 1/a + 1/b + 1/c = 1/1983 in positive integers.

2. An equilateral triangle ABC has side a. A square is constructed on the outside of each side of the triangle. A right regular pyramid with sloping side a is placed on each square. These pyramids are rotated about the sides of the triangle so that the apex of each pyramid comes to a common point above the triangle. Show that when this has been done the other vertices of the bases of the pyramids (apart from the vertices of the triangle) form a regular hexagon.

3. Show that 1 + 1/2 + 1/3 + ... + 1/n is not an integer for n > 1.

4. Show that it is possible to color each point of a circle red or blue so that no right-angled triangle inscribed in the circle has its vertices all the same color.

5. Show that $1 \le n^{1/n} \le 2$ for all positive integers n. Find the smallest k such that $1 \le n^{1/n} \le k$ for all positive integers n.

6. Show that the maximum number of spheres of radius 1 that can be placed touching a fixed sphere of radius 1 so that no pair of spheres has an interior point in common is between 12 and 14.

6th Brasil 1984

1. Find all solutions in positive integers to $(n+1)^k - 1 = n!$.

2. Each day 289 students are divided into 17 groups of 17. No two students are ever in the same group more than once. What is the largest number of days that this can be done?

3. Given a regular dodecahedron of side a. Take two pairs of opposite faces: E, E' and F, F'. For the pair E, E' take the line joining the centers of the faces and take points A and C on the line each a distance m outside one of the faces. Similarly, take B and D on the line joining the centers of F, F' each a distance m outside one of the faces. Show that ABCD is a rectangle and find the ratio of its side lengths.

4. ABC is a triangle with $\angle A = 90^o$. For a point D on the side BC, the feet of the perpendiculars to AB and AC are E and F. For which point D is EF a minimum?

5. ABCD is any convex quadrilateral. Squares center E, F, G, H are constructed on the outside of the edges AB, BC, CD and DA respectively. Show that EG and FH are equal and perpendicular.

6. There is a piece on each square of the solitaire board shown except for the central square. A move can be made when there are three adjacent squares in a horizontal or vertical line with two adjacent squares occupied and the third square vacant. The move is to remove the two pieces from the occupied squares and to place a piece on the third square. (One can regard one of the pieces as hopping over the other and taking it.) Is it possible to end up with a single piece on the board, on the square marked X?

7th Brasil 1985

1. a, b, c, d are integers with $ad \neq bc$. Show that $1/((ax+b)(cx+d))$ can be written in the form $r/(ax+b) + s/(cx+d)$. Find the sum $1/1\cdot4 + 1/4\cdot7 + 1/7\cdot10 + ... + 1/2998\cdot3001$.

2. Given n points in the plane, show that we can always find three which give an angle $\leq \pi/n$.

3. A convex quadrilateral is inscribed in a circle of radius 1. Show that the its perimeter less the sum of its two diagonals lies between 0 and 2.

4. a, b, c, d are integers. Show that $x^2 + ax + b = y^2 + cy + d$ has infinitely many integer solutions iff $a^2 - 4b = c^2 - 4d$.

5. A, B are reals. Find a necessary and sufficient condition for $Ax + B[x] = Ay + B[y]$ to have no solutions except $x = y$.

8th Brasil 1986

1. A ball moves endlessly on a circular billiard table. When it hits the edge it is reflected. Show that if it passes through a point on the table three times, then it passes through it infinitely many times.

2. Find the number of ways that a positive integer n can be represented as a sum of one or more consecutive positive integers.

3. The Poincare plane is a half-plane bounded by a line R. The lines are taken to be (1) the half-lines perpendicular to R, and (2) the semicircles with center on R. Show that given any line L and any point P not on L, there are infinitely many lines through P which do not intersect L. Show that if ABC is a triangle, then the sum of its angles lies in the interval $(0, \pi)$.

4. Find all 10 digit numbers $a_0a_1...a_9$ such that for each k, a_k is the number of times that the digit k appears in the number.

5. A number is written in each square of a chessboard, so that each number not on the border is the mean of the 4 neighboring numbers. Show that if the largest number is N, then there is a number equal to N in the border squares.

9th Brasil 1987

1. $p(x_1, x_2, ... , x_n)$ is a polynomial with integer coefficients. For each positive integer r, k(r) is the number of n-tuples $(a_1, a_2, ... , a_n)$ such that $0 \le a_i \le r-1$ and $p(a_1, a_2, ... , a_n)$ is prime to r. Show that if u and v are coprime then $k(u \cdot v) = k(u) \cdot k(v)$, and if p is prime then $k(p^s) = p^{n(s-1)} k(p)$.

2. Given a point p inside a convex polyhedron P. Show that there is a face F of P such that the foot of the perpendicular from p to F lies in the interior of F.

3. Two players play alternately. The first player is given a pair of positive integers (x_1, y_1). Each player must replace the pair (x_n, y_n) that he is given by a pair of non-negative integers (x_{n+1}, y_{n+1}) such that $x_{n+1} = \min(x_n, y_n)$ and $y_{n+1} = \max(x_n, y_n) - k \cdot x_{n+1}$ for some positive integer k. The first player to pass on a pair with $y_{n+1} = 0$ wins. Find for which values of x_1/y_1 the first player has a winning strategy.

4. Given points A_1 (x_1, y_1, z_1), A_2 (x_2, y_2, z_2), ... , A_n (x_n, y_n, z_n) let P (x, y, z) be the point which minimizes $\sum (|x - x_i| + |y - y_i| + |z - z_i|)$. Give an example (for each $n > 4$) of points A_i for which the point P lies outside the convex hull of the points A_i.

5. A and B wish to divide a cake into two pieces. Each wants the largest piece he can get. The cake is a triangular prism with the triangular faces horizontal. A chooses a point P on the top face. B then chooses a vertical plane through the point P to divide the cake. B chooses which piece to take. Which point P should A choose in order to secure as large a slice as possible?

10th Brasil 1988

1. Find all primes which can be written both as a sum of two primes and as a difference of two primes.

2. P is a fixed point in the plane. A, B, C are points such that PA = 3, PB = 5, PC = 7 and the area ABC is as large as possible. Show that P must be the orthocenter of ABC.

3. Let N be the natural numbers and $N' = N \cup \{0\}$. Find all functions $f:N \to N'$ such that $f(xy) = f(x) + f(y)$, $f(30) = 0$ and $f(x) = 0$ for all $x = 7 \bmod 10$.

4. Two triangles have the same incircle. Show that if a circle passes through five of the six vertices of the two triangles, then it also passes through the sixth.

5. A figure on a computer screen shows n points on a sphere, no four coplanar. Some pairs of points are joined by segments. Each segment is colored red or blue. For each point there is a key that switches the colors of all segments with that point as endpoint. For every three points there is a sequence of key presses that makes the three segments between them red. Show that it is possible to make all the segments on the screen red. Find the smallest number of key presses that can turn all the segments red, starting from the worst case.

11th Brasil 1989

1. The triangle vertices (0,0), (0,1), (2,0) is repeatedly reflected in the three lines AB, BC, CA where A is (0,0), B is (3,0), C is (0,3). Show that one of the images has vertices (24,36), (24,37) and (26,36).

2. n is a positive integer such that n(n+1)/3 is a square. Show that n is a multiple of 3, and n+1 and n/3 are squares.

3. Let Z be the integers. $f : Z \to Z$ is defined by $f(n) = n - 10$ for $n > 100$ and $f(n) = f(f(n+11))$ for $n \leq 100$. Find the set of possible values of f.

4. A and B play a game. Each has 10 tokens numbered from 1 to 10. The board is two rows of squares. The first row is numbered 1 to 1492 and the second row is numbered 1 to 1989. On the nth turn, A places his token number n on any empty square in either row and B places his token on any empty square in the other row. B wins if the order of the tokens is the same in the two rows, otherwise A wins. Which player has a winning strategy? Suppose each player has k tokens, numbered from 1 to k. Who has the winning strategy? Suppose that both rows are all the integers? Or both all the rationals?

5. The circumcenter of a tetrahedron lies inside the tetrahedron. Show that at least one of its edges is at least as long as the edge of a regular tetrahedron with the same circumsphere.

12th Brasil 1990

1. Show that a convex polyhedron with an odd number of faces has at least one face with an even number of edges.

2. Show that there are infinitely many positive integer solutions to $a^3 + 1990b^3 = c^4$.

3. Each face of a tetrahedron is a triangle with sides a, b, c and the tetrahedon has circumradius 1. Find $a^2 + b^2 + c^2$.

4. ABCD is a convex quadrilateral. E, F, G, H are the midpoints of sides AB, BC, CD, DA respectively. Find the point P such that area PHAE = area PEBF = area PFCG = area PGDH.

5. Given that $f(x) = (ax+b)/(cx+d)$, $f(0) \neq 0$, $f(f(0)) \neq 0$. Put $F(x) = f(...(f(x) ...)$ (where there are n fs). If $F(0) = 0$, show that $F(x) = x$ for all x where the expression is defined.

13th Brasil 1991

1. At a party every woman dances with at least one man, and no man dances with every woman. Show that there are men M and M' and women W and W' such that M dances with W, M' dances with W', but M does not dance with W', and M' does not dance with W.

2. P is a point inside the triangle ABC. The line through P parallel to AB meets AC at AC at A_0 and BC at B_0. Similarly, the line through P parallel to CA meets AB at A_1 and BC at C_1, and the line through P parallel to BC meets AB at B_2 and AC at C_2. Find the point P such that $A_0B_0 = A_1B_1 = A_2C_2$.

3. Given $k > 0$, the sequence $a_1, a_2, a_3, ...$ is defined by its first two members and $a_{n+2} = a_{n+1} + (k/n)a_n$. For which k can we write a_n as a polynomial in n? For which k can we write an+1/an = p(n)/q(n)?

4. Show that there is a number of the form 199...91 (with n 9s) with $n > 2$ which is divisible by 1991.

5. $P_0 = (1,0)$, $P_1 = (1,1)$, $P_2 = (0,1)$, $P_3 = (0,0)$. P_{n+4} is the midpoint of P_nP_{n+1}. Q_n is the quadrilateral $P_nP_{n+1}P_{n+2}P_{n+3}$. A_n is the interior of Q_n. Find $\cap_{n\geq 0}A_n$.

14th Brasil 1992

1. The polynomial $x^3 + px + q$ has three distinct real roots. Show that $p < 0$.

2. Show that there is a positive integer n such that the first 1992 digits of n^{1992} are 1.

3. Given positive real numbers $x_1, x_2, ... , x_n$ find the polygon $A_0A_1..A_n$ with $A_0A_1 = x_1$, $A_1A_2 = x_2, ... , A_{n-1}A_n = x_n$ which has greatest area.

4. ABC is a triangle. Find D on AC and E on AB such that area ADE = area DEBC and DE has minimum possible length.

5. Let d(n) be the number of positive divisors of n. Show that $n(1/2 + 1/3 + ... + 1/n) \le d(1) + d(2) + ... + d(n) \le n(1 + 1/2 + 1/3 + ... + 1/n)$.

6. Given a set of n elements, find the largest number of subsets such that no subset is contained in any other.

7. Find all solutions in positive integers to $n^a + n^b = n^c$.

8. In a chess tournament each player plays every other player once. A player gets 1 point for a win, ½ point for a draw and 0 for a loss. Both men and women played in the tournament and each player scored the same total of points against women as against men. Show that the total number of players must be a square.

9. Show that for each n > 5 it is possible to find a convex polyhedron with all faces congruent such that each face has another face parallel to it.

15th Brasil 1993

1. The sequence $a_1, a_2, a_3, ...$ is defined by $a_1 = 8$, $a_2 = 18$, $a_{n+2} = a_{n+1}a_n$. Find all terms which are perfect squares.

2. A real number with absolute value less than 1 is written in each cell of an n x n array, so that the sum of the numbers in each 2 x 2 square is zero. Show that for n odd the sum of all the numbers is less than n.

3. Given a circle and its center O, a point A inside the circle and a distance h, construct a triangle BAC with $\angle A = 90^o$, B and C on the circle and the altitude from A length h.

4. ABCD is a convex quadrilateral with $\angle BAC = 30^o$, $\angle CAD = 20^o$, $\angle ABD = 50^o$, $\angle DBC = 30^o$. If the diagonals intersect at P, show that PC = PD.

5. Find a real-valued function f(x) on the non-negative reals such that f(0) = 0, and f(2x+1) = 3f(x) + 5 for all x.

16th Brasil 1994

1. The edges of a cube are labeled from 1 to 12 in an arbitrary manner. Show that it is not possible to get the sum of the edges at each vertex the same. Show that we can get eight vertices with the same sum if one of the labels is changed to 13.

2. Given any convex polygon, show that there are three consecutive vertices such that the polygon lies inside the circle through them.

3. We are given n objects of identical appearance, but different mass, and a balance which can be used to compare any two objects (but only one object can be placed in each pan at a time). How many times must we use the balance to find the heaviest object and the lightest object?

4. Show that if the positive real numbers a, b satisfy $a^3 = a+1$ and $b^6 = b+3a$, then $a > b$.

5. Call a *super-integer* an infinite sequence of decimal digits: $...d_n...d_2d_1$. Given two such super-integers $...c_n...c_2c_1$ and $...d_n...d_2d_1$, their product $...p_n...p_2p_1$ is formed by taking $p_n...p_2p_1$ to be the last n digits of the product $c_n...c_2c_1$ and $d_n...d_2d_1$. Can we find two non-zero super-integers with zero product (a zero super-integer has all its digits zero).

6. A triangle has semi-perimeter s, circumradius R and inradius r. Show that it is right-angled iff $2R = s - r$.

17th Brasil 1995

A1. ABCD is a quadrilateral with a circumcircle center O and an inscribed circle center I. The diagonals intersect at S. Show that if two of O, I, S coincide, then it must be a square.

A2. Find all real-valued functions on the positive integers such that $f(x + 1019) = f(x)$ for all x, and $f(xy) = f(x) f(y)$ for all xy.

A3. Let p(n) be the largest prime which divides n. Show that there are infinitely many positive integers n such that $p(n) < p(n+1) < p(n+2)$.

B1. A regular tetrahedron has side L. What is the smallest x such that the tetrahedron can be passed through a loop of twine of length x?

B2. Show that the nth root of a rational (for n a positive integer) cannot be a root of the polynomial $x^5 - x^4 - 4x^3 + 4x^2 + 2$.

B3. X has n elements. F is a family of subsets of X each with three elements, such that any two of the subsets have at most one element in common. Show that there is a subset of X with at least $\sqrt{(2n)}$ members which does not contain any members of F.

18th Brasil 1996

A1. Show that the equation $x^2 + y^2 + z^2 = 3xyz$ has infinitely many solutions in positive integers.

A2. Does there exist a set of $n > 2$ points in the plane such that no three are collinear and the circumcenter of any three points of the set is also in the set?

A3. Let f(n) be the smallest number of 1s needed to represent the positive integer n using only 1s, + signs, x signs and brackets. For example, you could represent 80 with 13 1s as follows: (1+1+1+1+1)x(1+1+1+1)x(1+1+1+1). Show that $3 \log_3 n \le f(n) \le 5 \log_3 n$ for $n > 1$.

B1. ABC is acute-angled. D s a variable point on the side BC. O_1 is the circumcenter of ABD, O_2 is the circumcenter of ACD, and O is the circumcenter of AO_1O_2. Find the locus of O.

B2. There are n boys $B_1, B_2, \dots , B_n$ and n girls $G_1, G_2, \dots , G_n$. Each boy ranks the girls in order of preference, and each girl ranks the boys in order of preference. Show that we can arrange the boys and girls into n pairs so that we cannot find a boy and a girl who prefer each other to their partners. For example if (B_1, G_3) and (B_4, G_7) are two of the pairs, then it must not be the case that B_4 prefers G_3 to G_7 and G_3 prefers B_4 to B_1.

B3. Let p(x) be the polynomial $x^3 + 14x^2 - 2x + 1$. Let $p^n(x)$ denote $p(p^{n-1}(x))$. Show that there is an integer N such that $p^N(x) - x$ is divisible by 101 for all integers x.

19th Brasil 1997

A1. Given $R, r > 0$. Two circles are drawn radius R, r which meet in two points. The line joining the two points is a distance D from the center of one circle and a distance d from the center of the other. What is the smallest possible value for D+d?

A2. A is a set of n non-negative integers. We say it has *property P* if the set $\{x + y: x, y \in A\}$ has $n(n+1)/2$ elements. We call the largest element of A minus the smallest element, the *diameter* of A. Let f(n) be the smallest diameter of any set A with property P. Show that $n^2/4 \le f(n) < n^3$.

A3. Let R be the reals, show that there are no functions f, g: $R \to R$ such that $g(f(x)) = x^3$ and $f(g(x)) = x^2$ for all x. Let S be the set of all real numbers > 1. Show that there are functions f, g : $S \to S$ satsfying the condition above.

B1. Let F_n be the Fibonacci sequence $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$. Put $V_n = \sqrt{(F_n^2 + F_{n+2}^2)}$. Show that V_n, V_{n+1}, V_{n+2} are the sides of a triangle of area ½.

B2. c is a rational. Define $f^0(x) = x$, $f^{n+1}(x) = f(f^n(x))$. Show that there are only finitely many x such that the sequence $f^0(x)$, $f^1(x)$, $f^2(x)$, ... takes only finitely many values.

B3. f is a map on the plane such that two points a distance 1 apart are always taken to two points a distance 1 apart. Show that for any d, f takes two points a distance d apart to two points a distance d apart.

20th Brasil 1998

A1. 15 positive integers < 1998 are relatively prime (no pair has a common factor > 1). Show that at least one of them must be prime.

A2. ABC is a triangle. D is the midpoint of AB, E is a point on the side BC such that BE = 2 EC and $\angle ADC = \angle BAE$. Find $\angle BAC$.

A3. Two players play a game as follows. There n > 1 rounds and $d \ge 1$ is fixed. In the first round A picks a positive integer m_1, then B picks a positive integer $n_1 \ne m_1$. In round k (for k = 2, ... , n), A picks an integer m_k such that $m_{k-1} < m_k \le m_{k-1} + d$. Then B picks an integer n_k such that $n_{k-1} < n_k \le n_{k-1} + d$. A gets $\gcd(m_k,n_{k-1})$ points and B gets $\gcd(m_k,n_k)$ points. After n rounds, A wins if he has at least as many points as B, otherwise he loses. For each n, d which player has a winning strategy?

B1. Two players play a game as follows. The first player chooses two non-zero integers A and B. The second player forms a quadratic with A, B and 1998 as coefficients (in any order). The first player wins iff the equation has two distinct rational roots. Show that the first player can always win.

B2. Let N = {0, 1, 2, 3, ... }. Find all functions f : N → N which satisfy $f(2f(n)) = n + 1998$ for all n.

B3. Two mathematicians, lost in Berlin, arrived on the corner of Barbarossa street with Martin Luther street and need to arrive on the corner of Meininger street with Martin Luther street. Unfortunately they don't know which direction to go along Martin Luther Street to reach Meininger Street nor how far it is, so they must go fowards and backwards along Martin Luther street until they arrive on the desired corner. What is the smallest value for a positive integer K so that they can be sure that if there are N blocks between Barbarossa street and Meininger street then they can arrive at their destination by walking no more than KN blocks (no matter what N turns out to be)?

21st Brasil 1999

A1. ABCDE is a regular pentagon. The star ACEBD has area 1. AC and BE meet at P, BD and CE meet at Q. Find the area of APQD.

A2. Let d_n be the nth decimal digit of $\sqrt{2}$. Show that d_n cannot be zero for all of n = 1000001, 1000002, 1000003, ... , 3000000.

A3. How many pieces can be placed on a 10 x 10 board (each at the center of its square, at most one per square) so that no four pieces form a rectangle with sides parallel to the sides of the board?

B1. A spherical planet has finitely many towns. If there is a town at X, then there is also a town at X', the antipodal point. Some pairs of towns are connected by direct roads. No such roads cross (except at endpoints). If there is a direct road from A to B, then there is also a direct road from A' to B'. It is possible to get from any town to any other town by some sequence of roads. The populations of two towns linked by a direct road differ by at most 100. Show that there must be two antipodal towns whose populations differ by at most 100.

B2. n teams wish to play $n(n-1)/2$ games so that each team plays every other team just once. No team may play more than once per day. What is the minimum number of days required for the tournament?

B3. Given any triangle ABC, show how to construct A' on the side AB, B' on the side BC, C' on the side CA, so that ABC and A'B'C' are similar (with ∠A = ∠A', ∠B = ∠B' and ∠C = ∠C').

22nd Brasil 2000

A1. A piece of paper has top edge AD. A line L from A to the bottom edge makes an angle x with the line AD. We want to trisect x. We take B and C on the vertical ege through A such that AB = BC. We then fold the paper so that C goes to a point C' on the line L and A goes to a point A' on the horizontal line through B. The fold takes B to B'. Show that AA' and AB' are the required trisectors.

A2. Let s(n) be the sum of all positive divisors of n, so s(6) = 12. We say n is *almost perfect* if s(n) = 2n - 1. Let mod(n, k) denote the residue of n modulo k (in other words, the remainder of dividing n by k). Put t(n) = mod(n, 1) + mod(n, 2) + ... + mod(n, n). Show that n is almost perfect iff t(n) = t(n-1).

A3. Define f on the positive integers by $f(n) = k^2 + k + 1$, where 2^k is the highest power of 2 dividing n. Find the smallest n such that $f(1) + f(2) + ... + f(n) \geq 123456$.

B1. An infinite road has traffic lights at intervals of 1500m. The lights are all synchronised and are alternately green for 3/2 minutes and red for 1 minute. For which v can a car travel at a constant speed of v m/s without ever going through a red light?

B2. X is the set of all sequences $a_1, a_2, ... , a_{2000}$ such that each of the first 1000 terms is 0, 1 or 2, and each of the remaining terms is 0 or 1. The distance between two members a and b of X is defined as the number of i for which a_i and b_i are unequal. Find the number of functions $f : X \to X$ which preserve distance.

B3. C is a wooden cube. We cut along every plane which is perpendicular to the segment joining two distinct vertices and bisects it. How many pieces do we get?

23rd Brasil 2001

A1. Prove that $(a + b)(a + c) \geq 2(abc(a + b + c))^{1/2}$ for all positive reals.

A2. Given $a_0 > 1$, the sequence $a_0, a_1, a_2, ...$ is such that for all k > 0, a_k is the smallest integer greater than a_{k-1} which is relatively prime to all the earlier terms in the sequence. Find all a_0 for which all terms of the sequence are primes or prime powers.

A3. ABC is a triangle. The points E and F divide AB into thirds, so that AE = EF = FB. D is the foot of the perpendicular from E to the line BC, and the lines AD and CF are perpendicular. ∠ACF = 3 ∠BDF. Find DB/DC.

B1. A calculator treats angles as radians. It initially displays 1. What is the largest value that can be achieved by pressing the buttons cos or sin a total of 2001 times? (So you might press cos five times, then sin six times and so on with a total of 2001 presses.)

B2. An *altitude* of a convex quadrilateral is a line through the midpoint of a side perpendicular to the opposite side. Show that the four altitudes are concurrent iff the quadrilateral is cyclic.

B3. A one-player game is played as follows. There is bowl at each integer on the x-axis. All the bowls are initially empty, except for that at the origin, which contains n stones. A move is either (A) to remove two stones from a bowl and place one in each of the two adjacent bowls, or (B) to remove a stone from each of two adjacent bowls and to add one stone to the bowl immediately to their left. Show that only a finite number of moves can be made and that the final position (when no more moves are possible) is independent of the moves made (for given n).

24th Brasil 2002

A1. Show that there is a set of 2002 distinct positive integers such that the sum of one or more elements of the set is never a square, cube, or higher power.

A2. ABCD is a cyclic quadrilateral and M a point on the side CD such that ADM and ABCM have the same area and the same perimeter. Show that two sides of ABCD have the same length.

A3. The squares of an m x n board are labeled from 1 to mn so that the squares labeled i and i+1 always have a side in common. Show that for some k the squares k and k+3 have a side in common.

B1. For any non-empty subset A of {1, 2, ... , n} define f(A) as the largest element of A minus the smallest element of A. Find ∑ f(A) where the sum is taken over all non-empty subsets of {1, 2, ... , n}.

B2. A finite collection of squares has total area 4. Show that they can be arranged to cover a square of side 1.

B3. Show that we cannot form more than 4096 binary sequences of length 24 so that any two differ in at least 8 positions.

25th Brasil 2003

A1. Find the smallest positive prime that divides $n^2 + 5n + 23$ for some integer n.

A2. Let S be a set with n elements. Take a positive integer k. Let $A_1, A_2, \dots A_k$ be any distinct subsets of S. For each i take $B_i = A_i$ or $S - A_i$. Find the smallest k such that we can always choose B_i so that $\cup B_i = S$.

A3. ABCD is a parallelogram with perpendicular diagonals. Take points E, F, G, H on sides AB, BC, CD, DA respectively so that EF and GH are tangent to the incircle of ABCD. Show that EH and FG are parallel.

B1. Given a circle and a point A inside the circle, but not at its center. Find points B, C, D on the circle which maximise the area of the quadrilateral ABCD.

B2. f(x) is a real-valued function defined on the positive reals such that (1) if $x < y$, then $f(x) < f(y)$, (2) $f(2xy/(x+y)) \ge (f(x) + f(y))/2$ for all x. Show that $f(x) < 0$ for some value of x.

B3. A graph G with n vertices is called *great* if we can label each vertex with a different positive integer $\le [n^2/4]$ and find a set of non-negative integers D so that there is an edge between two vertices iff the difference between their labels is in D. Show that if n is sufficiently large we can always find a graph with n vertices which is not great.

✹***Part 3*** **Mexican Mathematical Olympiad(MMO)**

1st Mexican 1987

A1. a/b and c/d are positive fractions in their lowest terms such that $a/b + c/d = 1$. Show that $b = d$.

A2. How many positive integers divide 20! ?

A3. L and L' are parallel lines and P is a point midway between them. The variable point A lies L, and A' lies on L' so that $\angle APA' = 90^o$. X is the foot of the perpendicular from P to the line AA'. Find the locus of X as A varies.

A4. Let N be the product of all positive integers ≤ 100 which have exactly three positive divisors. Find N and show that it is a square.

B1. ABC is a triangle with $\angle A = 90^o$. M is a variable point on the side BC. P, Q are the feet of the perpendiculars from M to AB, AC. Show that the areas of BPM, MQC, AQMP cannot all be equal.

B2. Prove that $(n^3 - n)(5^{8n+4} + 3^{4n+2})$ is a multiple of 3804 for all positive integers n.

B3. Show that $n^2 + n - 1$ and $n^2 + 2n$ have no common factor.

B4. ABCD is a tetrahedron. The plane ABC is perpendicular to the line BD. $\angle ADB = \angle CDB = 45^o$ and $\angle ABC = 90^o$. Find $\angle ADC$. A plane through A perpendicular to DA meets the line BD at Q and the line CD at R. If AD = 1, find AQ, AR, and QR.

2nd Mexican 1988

A1. In how many ways can we arrange 7 white balls and 5 black balls in a line so that there is at least one white ball between any two black balls?

A2. If m and n are positive integers, show that 19 divides 11m + 2n iff it divides 18m + 5n.

A3. Two circles of different radius R and r touch externally. The three common tangents form a triangle. Find the area of the triangle in terms of R and r.

A4. How many ways can we find 8 integers $a_1, a_2, \dots , a_8$ such that $1 \leq a_1 \leq a_2 \leq \dots \leq a_8 \leq 8$?

B1. a and b are relatively prime positive integers, and n is an integer. Show that the greatest common divisor of a^2+b^2-nab and $a+b$ must divide $n+2$.

B2. B and C are fixed points on a circle. A is a variable point on the circle. Find the locus of the incenter of ABC as A varies.

B3. [unclear]

B4. Calculate the volume of an octahedron which has an inscribed sphere of radius 1.

3rd Mexican 1989

A1. The triangle ABC has AB = 5, the medians from A and B are perpendicular and the area is 18. Find the lengths of the other two sides.

A2. Find integers m and n such that n^2 is a multiple of m, m^3 is a multiple of n^2, n^4 is a multiple of m^3, m^5 is a multiple of n^4, but n^6 is not a multiple of m^5.

A3. Show that there is no positive integer of 1989 digits, at least three of them 5, such that the sum of the digits is the same as the product of the digits.

B1. Find a positive integer n with decimal expansion $a_ma_{m-1}...a_0$ such that $a_1a_0a_ma_{m-1}...a_20 = 2n$.

B2. C_1 and C_2 are two circles of radius 1 which touch at the center of a circle C of radius 2. C_3 is a circle inside C which touches C, C_1 and C_2. C_4 is a circle inside C which touches C, C_1 and C_3. Show that the centers of C, C_1, C_3 and C_4 form a rectangle.

B3. How many paths are there from A to B which do not pass through any vertex twice and which move only downwards or sideways, never up?

4th Mexican 1990

A1. How many paths are there from A to the line BC if the path does not go through any vertex twice and always moves to the left?

A2. ABC is a triangle with $\angle B = 90^o$ and altitude BH. The inradii of ABC, ABH, CBH are r, r_1, r_2. Find a relation between them.

A3. Show that $n^{n-1} - 1$ is divisible by $(n-1)^2$ for n > 2.

B1. Find 0/1 + 1/1 + 0/2 + 1/2 + 2/2 + 0/3 + 1/3 + 2/3 + 3/3 + 0/4 + 1/4 + 2/4 + 3/4 + 4/4

+ 0/5 + 1/5 + 2/5 + 3/5 + 4/5 + 5/5 + 0/6 + 1/6 + 2/6 + 3/6 + 4/6 + 5/6 + 6/6.

B2. Given 19 points in the plane with integer coordinates, no three collinear, show that we can always find three points whose centroid has integer coordinates.

B3. ABC is a triangle with $\angle C = 90^o$. E is a point on AC, and F is the midpoint of EC. CH is an altitude. I is the circumcenter of AHE, and G is the midpoint of BC. Show that ABC and IGF are similar.

5th Mexican 1991

A1. Find the sum of all positive irreducible fractions less than 1 whose denominator is 1991.

A2. n is palindromic (so it reads the same backwards as forwards, eg 15651) and n = 2 mod 3, n = 3 mod 4, n = 0 mod 5. Find the smallest such positive integer. Show that there are infinitely many such positive integers.

A3. 4 spheres of radius 1 are placed so that each touches the other three. What is the radius of the smallest sphere that contains all 4 spheres?

B1. ABCD is a convex quadrilateral with AC perpendicular to BD. M, N, R, S are the midpoints of AB, BC, CD, DA. The feet of the perpendiculars from M, N, R, S to CD, DA, AB, BC are W, X, Y, Z. Show that M, N, R, S, W, X, Y, Z lie on the same circle.

B2. The sum of the squares of two consecutive positive integers can be a square, for example $3^2 + 4^2 = 5^2$. Show that the sum of the squares of 3 or 6 consecutive positive integers cannot be a square. Give an example of the sum of the squares of 11 consecutive positive integers which is a square.

B3. Let T be a set of triangles whose vertices are all vertices of an n-gon. Any two triangles in T have either 0 or 2 common vertices. Show that T has at most n members.

6th Mexican 1992

A1. The tetrahedron OPQR has the $\angle POQ = \angle POR = \angle QOR = 90^o$. X, Y, Z are the midpoints of PQ, QR and RP. Show that the four faces of the tetrahedron OXYZ have equal area.

A2. Given a prime number p, how many 4-tuples (a, b, c, d) of positive integers with $0 < a, b, c, d < p-1$ satisfy $ad = bc \mod p$?

A3. Given 7 points inside or on a regular hexagon, show that three of them form a triangle with area ≤ 1/6 the area of the hexagon.

B1. Show that $1 + 11^{11} + 111^{111} + 1111^{1111} + ... + 1111111111^{1111111111}$ is divisible by 100.

B2. x, y, z are positive reals with sum 3. Show that $6 < \sqrt{2x+3} + \sqrt{2y+3} + \sqrt{2z+3} < 3\sqrt{5}$.

B3. ABCD is a rectangle. I is the midpoint of CD. BI meets AC at M. Show that the line DM passes through the midpoint of BC. E is a point outside the rectangle such that AE = BE and $\angle AEB = 90^o$. If BE = BC = x, show that EM bisects $\angle AMB$. Find the area of AEBM in terms of x.

7th Mexican 1993

A1. ABC is a triangle with $\angle A = 90^o$. Take E such that the triangle AEC is outside ABC and AE = CE and $\angle AEC = 90^o$. Similarly, take D so that ADB is outside ABC and similar to AEC. O is the midpoint of BC. Let the lines OD and EC meet at D', and the lines OE and BD meet at E'. Find area DED'E' in terms of the sides of ABC.

A2. Find all numbers between 100 and 999 which equal the sum of the cubes of their digits.

A3. Given a pentagon of area 1993 and 995 points inside the pentagon, let S be the set containing the vertices of the pentagon and the 995 points. Show that we can find three points of S which form a triangle of area ≤ 1.

B1. f(n,k) is defined by (1) f(n,0) = f(n,n) = 1 and (2) f(n,k) = f(n-1,k-1) + f(n-1,k) for $0 < k < n$. How many times do we need to use (2) to find f(3991,1993)?

B2. OA, OB, OC are three chords of a circle. The circles with diameters OA, OB meet again at Z, the circles with diameters OB, OC meet again at X, and the circles with diameters OC, OA meet again at Y. Show that X, Y, Z are collinear.

B3. p is an odd prime. Show that p divides n(n+1)(n+2)(n+3) + 1 for some integer n iff p divides m^2 - 5 for some integer m.

8th Mexican 1994

A1. The sequence 1, 2, 4, 5, 7, 9 ,10, 12, 14, 16, 17, ... is formed as follows. First we take one odd number, then two even numbers, then three odd numbers, then four even numbers, and so on. Find the number in the sequence which is closest to 1994.

A2. The 12 numbers on a clock face are rearranged. Show that we can still find three adjacent numbers whose sum is 21 or more.

A3. ABCD is a parallelogram. Take E on the line AB so that BE = BC and B lies between A and E. Let the line through C perpendicular to BD and the line through E perpendicular to AB meet at F. Show that ∠DAF = ∠BAF.

B1. A capricious mathematician writes a book with pages numbered from 2 to 400. The pages are to be read in the following order. Take the last unread page (400), then read (in the usual order) all pages which are not relatively prime to it and which have not been read before. Repeat until all pages are read. So, the order would be 2, 4, 5, ... , 400, 3, 7, 9, ... , 399, ... What is the last page to be read?

B2. ABCD is a convex quadrilateral. Take the 12 points which are the feet of the altitudes in the triangles ABC, BCD, CDA, DAB. Show that at least one of these points must lie on the sides of ABCD.

B3. Show that we cannot tile a 10 x 10 board with 25 pieces of type A, or with 25 pieces of type B, or with 25 pieces of type C.

9th Mexican 1995

A1. N students are seated at desks in an m x n array, where m, n ≥ 3. Each student shakes hands with the students who are adjacent horizontally, vertically or diagonally. If there are 1020 handshakes, what is N?

A2. 6 points in the plane have the property that 8 of the distances between them are 1. Show that three of the points form an equilateral triangle with side 1.

A3. A, B, C, D are consecutive vertices of a regular 7-gon. AL and AM are tangents to the circle center C radius CB. N is the point of intersection of AC and BD. Show that L, M, N are collinear.

B1. Find 26 elements of $\{1, 2, 3, ..., 40\}$ such that the product of two of them is never a square. Show that one cannot find 27 such elements.

B2. ABCDE is a convex pentagon such that the triangles ABC, BCD, CDE, DEA and EAB have equal area. Show that (1/4) area ABCDE < area ABC < (1/3) area ABCDE.

B3. A 1 or 0 is placed on each square of a 4 x 4 board. One is allowed to change each symbol in a row, or change each symbol in a column, or change each symbol in a diagonal (there are 14 diagonals of lengths 1 to 4). For which arrangements can one make changes which end up with all 0s?

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10th Mexican 1996

A1. ABCD is a quadrilateral. P and Q are points on the diagonal BD such that the points are in the order B, P, Q, D and BP = PQ = QD. The line AP meets BC at E, and the line Q meets CD at F. Show that ABCD is a parallelogram iff E and F are the midpoints of their sides.

A2. 64 tokens are numbered 1, 2, ... , 64. The tokens are arranged in a circle around 1996 lamps which are all turned off. Each minute the tokens are all moved. Token number n is moved n places clockwise. More than one token is allowed to occupy the same place. After each move we count the number of tokens which occupy the same place as token 1 and turn on that number of lamps. Where is token 1 when the last lamp is turned on?

A3. Show that it is not possible to cover a 6 x 6 board with 1 x 2 dominos so that each of the 10 lines of length 6 that form the board (but do not lie along its border) bisects at least one domino. But show that we can cover a 5 x 6 board with 1 x 2 dominos so that each of the 9 lines of length 5 or 6 that form the board (but do not lie along its border) bisects at least one domino.

B1. For which n can we arrange the numbers 1, 2, 3, ... , 16 in a 4 x 4 array so that the eight row and column sums are all distinct and all multiples of n?

B2. Arrange the numbers 1, 2, 3, ... , n^2 in order in a n x n array (so that the first row is 1, 2, 3, ... , n, the second row is n+1, n+2, ... , 2n, and so on). For each path from 1 to n^2 which consists entirely of steps to the right and steps downwards, find the sum of the numbers in the path. Let M be the largest such sum and m the smallest. Show that M - m is a cube and that we cannot get the sum 1996 for a square of any size.

B3. ABC is an acute-angled triangle with AB < BC < AC. The points A', B', C' are such that AA' is perpendicular to BC and has the same length. Similarly, BB' is perpendicular to AC and has the same length, and CC' is perpendicular to AB and has the same length. The orthocenter H of ABC and A' are on the same side of A. Similarly, H and B' are on the same side of B, and H and C' are on the same side of C. Also $\angle AC'B = 90^o$. Show that A', B', C' are collinear.

11th Mexican 1997

A1. Find all primes p such that $8p^4 - 3003$ is a (positive) prime.

A2. ABC is a triangle with centroid G. P, P' are points on the side BC, Q is a point on the side AC, R is a point on the side AB, such that AR/RB = BP/PC = CQ/QA = CP'/P'B. The lines AP' and QR meet at K. Show that P, G and K are collinear.

A3. Show that it is possible to place the numbers 1, 2, ... , 16 on the squares of a 4 x 4 board (one per square), so that the numbers on two squares which share a side differ by at most 4. Show that it is not possible to place them so that the difference is at most 3.

B1. 3 non-collinear points in space determine a unique plane, which contains the points. What is the smallest number of planes determined by 6 points in space if no three points are collinear and the points do not all lie in the same plane?

B2. ABC is a triangle. P, Q, R are points on the sides BC, CA, AB such that BQ, CR meet at A', CR, AP meet at B', AP, BQ meet at C' and we have AB' = B'C', BC' = C'A', CA' = A'B'. Find area PQR/area ABC.

B3. Show that we can represent 1 as $1/5 + 1/a_1 + 1/a_2 + ... + 1/a_n$ (for positive integers a_i) in infinitely many different ways.

12th Mexican 1998

A1. Given a positive integer we can take the sum of the squares of its digits. If repeating this operation a finite number of times gives 1 we call the number *tame*. Show that there are infinitely many pairs (n, n+1) of consecutive tame integers.

A2. The lines L and L' meet at A. P is a fixed point on L. A variable circle touches L at P and meets L' at Q and R. The bisector of ∠QPR meets the circle again at T. Find the locus of T as the circle varies.

A3. Each side and diagonal of an octagon is colored red or black. Show that there are at least 7 triangles whose vertices are vertices of the octagon and whose sides are the same color.

B1. Find all positive integers that can be written as $1/a_1 + 2/a_2 + ... + 9/a_9$, where a_i are positive integers.

B2. AB, AC are the tangents from A to a circle. Q is a point on the segment AC. The line BQ meets the circle again at P. The line through Q parallel to AB meets BC at J. Show that PJ is parallel to AC iff $BC^2 = AC \cdot QC$.

B3. Given 5 points, no 4 in the same plane, how many planes can be equidistant from the points? (A plane is equidistant from the points if the perpendicular distance from each point to the plane is the same.)

13th Mexican 1999

A1. 1999 cards are lying on a table. Each card has a red side and a black side and can be either side up. Two players play alternately. Each player can remove any number of cards showing the same color from the table or turn over any number of cards of the same color. The winner is the player who removes the last card. Does the first or second player have a winning strategy?

A2. Show that there is no arithmetic progression of 1999 distinct positive primes all less than 12345.

A3. P is a point inside the triangle ABC. D, E, F are the midpoints of AP, BP, CP. The lines BF, CE meet at L; the lines CD, AF meet at M; and the lines AE, BD meet at N. Show that area DNELFM = (1/3) area ABC. Show that DL, EM, FN are concurrent.

B1. 10 squares of a chessboard are chosen arbitrarily and the center of each chosen square is marked. The side of a square of the board is 1. Show that either two of the marked points are a distance $\leq \sqrt{2}$ apart or that one of the marked points is a distance 1/2 from the edge of the board.

B2. ABCD has AB parallel to CD. The exterior bisectors of $\angle B$ and $\angle C$ meet at P, and the exterior bisectors of $\angle A$ and $\angle D$ meet at Q. Show that PQ is half the perimeter of ABCD.

B3. A polygon has each side integral and each pair of adjacent sides perpendicular (it is not necessarily convex). Show that if it can be covered by non-overlapping 2 x 1 dominos, then at least one of its sides has even length.

14th Mexican 2000

A1. A, B, C, D are circles such that A and B touch externally at P, B and C touch externally at Q, C and D touch externally at R, and D and A touch externally at S. A does not intersect C, and B does not intersect D. Show that PQRS is cyclic. If A and C have radius 2, B and D have radius 3, and the distance between the centers of A and C is 6, find area PQRS.

A2. A triangle is constructed like that below, but with 1, 2, 3, ... , 2000 as the first row. Each number is the sum of the two numbers immediately above. Find the number at the bottom of the triangle.

```
1   2   3   4   5
  3   5   7   9
    8   12  16
      20  28
        48
```

A3. If A is a set of positive integers, take the set A' to be all elements which can be written as $\pm a_1 \pm a_2 \ldots \pm a_n$, where a_i are distinct elements of A. Similarly, form A" from A'. What is the smallest set A such that A" contains all of 1, 2, 3, ... , 40?

B1. Given positive integers a, b (neither a multiple of 5) we construct a sequence as follows: $a_1 = 5$, $a_{n+1} = a\, a_n + b$. What is the largest number of primes that can be obtained before the first composite member of the sequence?

B2. Given an n x n board with squares colored alternately black and white like a chessboard. An allowed move is to take a rectangle of squares (with one side greater than one square, and both sides odd or both sides even) and change the color of each square in the rectangle. For which n is it possible to end up with all the squares the same color by a sequence of allowed moves?

B3. ABC is a triangle with $\angle B > 90^o$. H is a point on the side AC such that AH = BH and BH is perpendicular to BC. D, E are the midpoints of AB, BC. The line through H parallel to AB meets DE at F. Show that $\angle BCF = \angle ACD$.

15th Mexican 2001

A1. Find all 7-digit numbers which are multiples of 21 and which have each digit 3 or 7.

A2. Given some colored balls (at least three different colors) and at least three boxes. The balls are put into the boxes so that no box is empty and we cannot find three balls of different colors which are in three different boxes. Show that there is a box such that all the balls in all the other boxes have the same color.

A3. ABCD is a cyclic quadrilateral. M is the midpoint of CD. The diagonals meet at P. The circle through P which touches CD at M meets AC again at R and BD again at Q.

The point S on BD is such that BS = DQ. The line through S parallel to AB meets AC at T. Show that AT = RC.

B1. For positive integers n, m define f(n,m) as follows. Write a list of 2001 numbers a_i, where $a_1 = m$, and a_{k+1} is the residue of a_k^2 mod n (for k = 1, 2, ... , 2000). Then put f(n,m) = $a_1 - a_2 + a_3 - a_4 + a_5 - ... + a_{2001}$. For which $n \geq 5$ can we find m such that $2 \leq m \leq n/2$ and f(m,n) > 0?

B2. ABC is a triangle with AB < AC and $\angle A = 2 \angle C$. D is the point on AC such that CD = AB. Let L be the line through B parallel to AC. Let L meet the external bisector of $\angle A$ at M and the line through C parallel to AB at N. Show that MD = ND.

B3. A collector of rare coins has coins of denominations 1, 2, ... , n (several coins for each denomination). He wishes to put the coins into 5 boxes so that: (1) in each box there is at most one coin of each denomination; (2) each box has the same number of coins and the same denomination total; (3) any two boxes contain all the denominations; (4) no denomination is in all 5 boxes. For which n is this possible?

16th Mexican 2002

A1. The numbers 1 to 1024 are written one per square on a 32 x 32 board, so that the first row is 1, 2, ... , 32, the second row is 33, 34, ... , 64 and so on. Then the board is divided into four 16 x 16 boards and the position of these boards is moved round clockwise, so that

```
AB goes to DA
DC         CB
```

then each of the 16 x 16 boards is divided into four equal 8 x 8 parts and each of these is moved around in the same way (within the 16 x 16 board). Then each of the 8 x 8 boards is divided into four 4 x 4 parts and these are moved around, then each 4 x 4 board is divided into 2 x 2 parts which are moved around, and finally the squares of each 2 x 2 part are moved around. What numbers end up on the main diagonal (from the top left to bottom right)?

A2. ABCD is a parallelogram. K is the circumcircle of ABD. The lines BC and CD meet K again at E and F. Show that the circumcenter of CEF lies on K.

A3. Does n^2 have more divisors = 1 mod 4 or = 3 mod 4?

B1. A domino has two numbers (which may be equal) between 0 and 6, one at each end. The domino may be turned around. There is one domino of each type, so 28 in all. We want to form a chain in the usual way, so that adjacent dominos have the same number at the adjacent ends. Dominos can be added to the chain at either end. We want to form the chain so that after each domino has been added the total of all the numbers is odd. For example, we could place first the domino (3,4), total 3 + 4 = 7. Then (1,3), total 1 + 3 + 3 + 4 = 11, then (4,4), total 11 + 4 + 4 = 19. What is the largest number of dominos that can be placed in this way? How many maximum-length chains are there?

B2. A *trio* is a set of three distinct integers such that two of the numbers are divisors or multiples of the third. Which trio contained in {1, 2, ... , 2002} has the largest possible sum? Find all trios with the maximum sum.

B3. ABCD is a quadrilateral with $\angle A = \angle B = 90^o$. M is the midpoint of AB and $\angle CMD = 90^o$. K is the foot of the perpendicular from M to CD. AK meets BD at P, and BK meets AC at Q. Show that $\angle AKB = 90^o$ and KP/PA + KQ/QB = 1.

17th Mexican 2003

A1. Find all positive integers with two or more digits such that if we insert a 0 between the units and tens digits we get a multiple of the original number.

A2. A, B, C are collinear with B betweeen A and C. K_1 is the circle with diameter AB, and K_2 is the circle with diameter BC. Another circle touches AC at B and meets K_1 again at P and K_2 again at Q. The line PQ meets K_1 again at R and K_2 again at S. Show that the lines AR and CS meet on the perpendicular to AC at B.

A3. At a party there are n women and n men. Each woman likes r of the men, and each man likes r of then women. For which r and s must there be a man and a woman who like each other?

B1. The quadrilateral ABCD has AB parallel to CD. P is on the side AB and Q on the side CD such that AP/PB = DQ/CQ. M is the intersection of AQ and DP, and N is the intersection of PC and QB. Find MN in terms of AB and CD.

B2. Some cards each have a pair of numbers written on them. There is just one card for each pair (a,b) with $1 \le a < b \le 2003$. Two players play the following game. Each removes a card in turn and writes the product ab of its numbers on the blackboard. The first player who causes the greatest common divisor of the numbers on the blackboard to fall to 1 loses. Which player has a winning strategy?

B3. Given a positive integer n, an allowed move is to form 2n+1 or 3n+2. The set S_n is the set of all numbers that can be obtained by a sequence of allowed moves starting with n. For example, we can form 5 → 11 → 35 so 5, 11 and 35 belong to S_5. We call m and n *compatible* if $S_m \cap S_n$ is non-empty. Which members of {1, 2, 3, ... , 2002} are compatible with 2003?

✹*Part 4* Indian Mathematical Olympiad(INMO)

INMO 1995

1. ABC is an acute-angled triangle with $\angle A = 30^o$. H is the orthocenter and M is the midpoint of BC. T is a point on HM such that HM = MT. Show that AT = 2 BC.

2. Show that there are infinitely many pairs (a,b) of coprime integers (which may be negative, but not zero) such that $x^2 + ax + b = 0$ and $x^2 + 2ax + b$ have integral roots.

3. Show that more 3 element subsets of {1, 2, 3, ... , 63} have sum greater than 95 than have sum less than 95.

4. ABC is a triangle with incircle K, radius r. A circle K', radius r', lies inside ABC and touches AB and AC and touches K externally. Show that $r'/r = \tan^2((\pi-A)/4)$.

5. $x_1, x_2, ... , x_n$ are reals > 1 such that $|x_i - x_{i+1}| < 1$ for i < n. Show that $x_1/x_2 + x_2/x_3 + ... + x_{n-1}/x_n + x_n/x_1 < 2n-1$.

6. Find all primes p for which $(2^{p-1} - 1)/p$ is a square.

INMO 1996

1. Given any positive integer n, show that there are distinct positive integers a, b such that a + k divides b + k for k = 1, 2, ... , n. If a, b are positive integers such that a + k divides b + k for all positive integers k, show that a = b.

2. C, C' are concentric circles with radii R, 3R respectively. Show that the orthocenter of any triangle inscribed in C must lie inside the circle C'. Conversely, show that any point inside C' is the orthocenter of some circle inscribed in C.

3. Find reals a, b, c, d, e such that $3a = (b + c + d)^3$, $3b = (c + d + e)^3$, $3c = (d + e + a)^3$, $3d = (e + a + b)^3$, $3e = (a + b + c)^3$.

4. X is a set with n elements. Find the number of triples (A, B, C) such that A, B, C are subsets of X, A is a subset of B, and B is a proper subset of C.

5. The sequence $a_1, a_2, a_3, \ldots$ is defined by $a_1 = 1$, $a_2 = 2$, $a_{n+2} = 2a_{n+1} - a_n + 2$. Show that for any m, $a_m a_{m+1}$ is also a term of the sequence.

6. A 2n x 2n array has each entry 0 or 1. There are just 3n 0s. Show that it is possible to remove all the 0s by deleting n rows and n columns.

INMO 1997

1. ABCD is a parallelogram. A line through C does not pass through the interior of ABCD and meets the lines AB, AD at E, F respectively. Show that $AC^2 + CE\cdot CF = AB\cdot AE + AD\cdot AF$.

2. Show that there do not exist positive integers m, n such that $m/n + (n+1)/m = 4$.

3. a, b, c are distinct reals such that $a + 1/b = b + 1/c = c + 1/a = t$ for some real t. Show that $t = -abc$.

4. In a unit square, 100 segments are drawn from the center to the perimeter, dividing the square into 100 parts. If all parts have equal perimeter p, show that $1.4 < p < 1.5$.

5. Find the number of 4 x 4 arrays with entries from {0, 1, 2, 3} such that the sum of each row is divisible by 4, and the sum of each column is divisible by 4.

6. a, b are positive reals such that the cubic $x^3 - ax + b = 0$ has all its roots real. α is the root with smallest absolute value. Show that $b/a < \alpha \le 3b/2a$.

INMO 1998

1. C is a circle with center O. AB is a chord not passing through O. M is the midpoint of AB. C' is the circle diameter OM. T is a point on C'. The tangent to C' at T meets C at P. Show that $PA^2 + PB^2 = 4 PT^2$.

2. a, b are positive rationals such that $a^{1/3} + b^{1/3}$ is also a rational. Show that $a^{1/3}$ and $b^{1/3}$ are rational.

3. p, q, r, s are integers and s is not a multiple of 5. If there is an integer a such that $pa^3 + qa^2 + ra + s$ is a multiple of 5, show that there is an integer b such that $sb^3 + rb^2 + qb + p$ is a multiple of 5.

4. ABCD is a cyclic quadrilateral inscribed in a circle radius 1. If $AB \cdot BC \cdot CD \cdot DA \geq 4$, show that ABCD is a square.

5. The quadratic $x^2 - (a+b+c)x + (ab+bc+ca) = 0$ has non-real roots. Show that a, b, c, are all positive and that there is a triangle with sides $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$.

6. $a_1, a_2, \ldots , a_{2n}$ is a sequence with two copies each of 0, 1, 2, ... , n-1. A subsequence of n elements is chosen so that its arithmetic mean is integral and as small as possible. Find this minimum value.

INMO 1999

1. ABC is an acute-angled triangle. AD is an altitude, BE a median, and CF an angle bisector. CF meets AD at M, and DE at N. FM = 2, MN = 1, NC = 3. Find the perimeter of ABC.

2. A rectangular field with integer sides and perimeter 3996 is divided into 1998 equal parts, each with integral area. Find the dimensions of the field.

3. Show that x5 + 2x + 1 cannot be factorised into two polynomials with integer coefficients (and degree ≥ 1).

4. X, X' are concentric circles. ABC, A'B'C' are equilateral triangles inscribed in X, X' respectively. P, P' are points on the perimeters of X, X' respectively. Show that $P'A^2 + P'B^2 + P'C^2 = A'P^2 + B'P^2 + C'P^2$.

5. Given any four distinct reals, show that we can always choose three A, b, C, such that the equations $ax^2 + x + b = 0$, $bx^2 + x + c = 0$, $cx^2 + x + a = 0$ either all have real roots, or all have non-real roots.

6. For which n can {1, 2, 3, ... , 4n} be divided into n disjoint 4-element subsets such that for each subset one element is the arithmetic mean of the other three?

INMO 2000

1. The incircle of ABC touches BC, CA, AB at K, L, M respectively. The line through A parallel to LK meets MK at P, and the line through A parallel to MK meets LK at Q. Show that the line PQ bisects AB and bisects AC.

2. Find the integer solutions to $a + b = 1 - c$, $a^3 + b^3 = 1 - c^2$.

3. a, b, c are non-zero reals, and x is real and satisfies $[bx + c(1-x)]/a = [cx + a(1-x)]/b = [ax + b(1-x)]/b$. Show that $a = b = c$.

4. In a convex quadrilateral PQRS, PQ = RS, $SP = (\sqrt{3} + 1)QR$, and $\angle RSP - \angle SQP = 30^o$. Show that $\angle PQR - \angle QRS = 90^o$.

5. a, b, c are reals such that $0 \le c \le b \le a \le 1$. Show that if α is a root of $z^3 + az^2 + bz + c = 0$, then $|\alpha| \le 1$.

6. Let f(n) be the number of incongruent triangles with integral sides and perimeter n, eg f(3) = 1, f(4) = 0, f(7) = 2. Show that f(1999) > f(1996) and f(2000) = f(1997).

INMO 2001

1. ABC is a triangle which is not right-angled. P is a point in the plane. A', B', C' are the reflections of P in BC, CA, AB. Show that [incomplete].

2. Show that $a^2 + b^2 + c^2 = (a-b)(b-c)(c-a)$ has infinitely many integral solutions.

3. a, b, c are positive reals with product 1. Show that $a^{b+c}b^{c+a}c^{a+b} \le 1$.

4. Show that given any nine integers, we can find four, a, b, c, d such that a + b - c - d is divisible by 20. Show that this is not always true for eight integers.

5. ABC is a triangle. M is the midpoint of BC. $\angle MAB = \angle C$, and $\angle MAC = 15^{o}$. Show that $\angle AMC$ is obtuse. If O is the circumcenter of ADC, show that AOD is equilateral.

6. Find all real-valued functions f on the reals such that f(x+y) = f(x) f(y) f(xy) for all x, y.

INMO 2002

1. ABCDEF is a convex hexagon. Consider the following statements. (1) AB is parallel to DE, (2) BC is parallel to EF, (3) CD is parallel to FA, (4) AE = BD, (5) BF = CE, (6) CA = DF. Show that if any five of these statements are true then the hexagon is cyclic.

2. Find the smallest positive value taken by $a^3 + b^3 + c^3 - 3abc$ for positive integers a, b, c. Find all a, b, c which give the smallest value.

3. x, y are positive reals such that x + y = 2. Show that $x^3y^3(x^3 + y^3) \le 2$.

4. Do there exist 100 lines in the plane, no three concurrent, such that they intersect in exactly 2002 points?

5. Do there exist distinct positive integers a, b, c such that a, b, c, -a+b+c, a-b+c, a+b-c, a+b+c form an arithmetic progression (in some order).

6. The numbers 1, 2, 3, ... , n^2 are arranged in an n x n array, so that the numbers in each row increase from left to right, and the numbers in each column increase from top to bottom. Let a_{ij} be the number in position i, j. Let b_j be the number of possible value for a_{jj}. Show that $b_1 + b_2 + ... + b_n = n(n^2-3n+5)/3$.

INMO 2003

1. ABC is acute-angled. P is an interior point. The line BP meets AC at E, and the line CP meets AB at F. AP meets EF at D. K is the foot of the perpendicular from D to BC. Show that KD bisects $\angle EKF$.

2. Find all primes p, q and even n > 2 such that $p^n + p^{n-1} + ... + p + 1 = q^2 + q + 1$.

3. Show that $8x^4 - 16x^3 + 16x^2 - 8x + k = 0$ has at least one real root for all real k. Find the sum of the non-real roots.

4. Find all 7-digit numbers which use only the digits 5 and 7 and are divisible by 35.

5. ABC has sides a, b, c. The triangle A'B'C' has sides a + b/2, b + c/2, c + a/2. Show that its area is at least (9/4) area ABC.

6. Each lottery ticket has a 9-digit numbers, which uses only the digits 1, 2, 3. Each ticket is colored red, blue or green. If two tickets have numbers which differ in all nine places, then the tickets have different colors. Ticket 122222222 is red, and ticket 222222222 is green. What color is ticket 123123123?

INMO 2004

1. ABCD is a convex quadrilateral. K, L, M, N are the midpoints of the sides AB, BC, CD, DA. BD bisects KM at Q. QA = QB = QC = QD, and LK/LM = CD/CB. Prove that ABCD is a square.

2. $p > 3$ is a prime. Find all integers a, b, such that $a^2 + 3ab + 2p(a+b) + p^2 = 0$.

3. If α is a real root of $x^5 - x^3 + x - 2 = 0$, show that $[\alpha^6] = 3$.

4. ABC is a triangle, with sides a, b, c (as usual), circumradius R, and exradii r_a, r_b, r_c. If $2R \le r_a$, show that $a > b$, $a > c$, $2R > r_b$, and $2R > r_c$.

5. S is the set of all (a, b, c, d, e, f) where a, b, c, d, e, f are integers such that $a^2 + b^2 + c^2 + d^2 + e^2 = f^2$. Find the largest k which divides abcdef for all members of S.

6. Show that the number of 5-tuples (a, b, c, d, e) such that abcde = 5(bcde + acde + abde + abce + abcd) is odd.

✸ ***Part 5*** **Astrian-Polish Mathematical Olympiad(AMO)**

19th Austrian-Polish 1996

1. Show that there are 3^{k-1} positive integers n such that n has k digits, all odd, n is divisible by 5 and n/5 has k odd digits. [For example, for k = 2, the possible numbers are 55, 75 and 95.]

2. ABCDEF is a convex hexagon is such that opposite sides are parallel and the perpendicular distance between each pair of opposite sides is equal. The angles at A and D are 90^o. Show that the diagonals BE and CF are at 45^o.

3. The polynomials $p_n(x)$ are defined by $p_0(x) = 0$, $p_1(x) = x$, $p_{n+2}(x) = x\, p_{n+1}(x) + (1 - x)\, p_n(x)$. Find the real roots of each $p_n(x)$.

4. The real numbers w, x, y, z have zero sum and sum of squares 1. Show that the sum $wx + xy + yz + zw$ lies between -1 and 0.

5. P is a convex polyhedron. S is a sphere which meets each edge of P at the two points which divide the edge into three equal parts. Show that there is a sphere which touches every edge of P.

6. k, n are positive integers such that n > k > 1. Find all real solutions $x_1, x_2, \dots, x_n$ to $x_i^3(x_i^2 + x_{i+1}^2 + \dots + x_{i+k-1}^2) = x_{i-1}^2$ for i = 1, 2, ... , n. [Note that we take x_0 to mean x_n, and x_{n+j} to mean x_j.]

7. Show that there are no non-negative integers m, n satisfying $m! + 48 = 48 (m + 1)^n$.

8. Show that there is no real polynomial of degree 998 such that $p(x)^2 - 1 = p(x^2 + 1)$ for all x.

9. A *block* is a rectangular parallelepiped with integer sides a, b, c which is not a cube. N blocks are used to form a 10 x 10 x 10 cube. The blocks may be different sizes. Show that if N ≥ 100, then at least two of the blocks must have the same dimensions and be placed with corresponding edges parallel. Prove the same for some number smaller than 100.

20th Austrian-Polish 1997

1. Four circles, none of which lies inside another, pass through the point P. Two circles touch the line L at P and the other two touch the line M at P. The other points of intersection of the circles are A, B, C, D. Show that A, B, C, D lie on a circle iff L and M are perpendicular.

2. A piece is on each square of an m x n board. The allowed move for each piece is h squares parallel to the bottom edge of the board and k squares parallel to the sides. How many ways can we move every piece simultaneously so that after the move there is still one piece on each square?

3. The 97 numbers 49/1, 49/2, 49/3, ... , 49/97 are written on a blackboard. We repeatedly pick two numbers a, b on the board and replace them by 2ab - a - b + 1 until only one number remains. What are the possible values of the final number?

4. ABCD is a convex quadrilateral with AB parallel to CD. The diagonals meet at E. X is the midpoint of the line joining the orthocenters of BEC and AED. Show that X lies on the perpendicular to AB through E.

5. Show that no cubic with integer coefficients can take the value ±3 at each of four distinct primes.

6. Show that there is no integer-valued function on the integers such that f(m + f(n)) = f(m) - n for all m, n.

7. Show that $x^2 + y^2 + 1 > x(y + 1)$ for all reals x, y. Find the largest k such that $x^2 + y^2 + 1 \geq kx(y + 1)$ for all reals x, y. Find the largest k such that $m^2 + n^2 + 1 \geq km(n + 1)$ for all integers m, n.

8. Let X be a set with n members. Find the largest number of subsets of X each with 3 members so that no two are disjoint.

9. k > 0 and P is a solid parallelepiped. S is the set of all points X for which there is a point Y in P such that XY ≤ k. Show that the volume of $S = V + Fk + \pi Ek^2/4 + 4\pi k^3/3$, where V, F, E are respectively the volume, surface area and total edge length of P.

21st Austrian-Polish 1998

1. Show that $(wx + yz - 1)^2 \geq (w^2 + y^2 - 1)(x^2 + z^2 - 1)$ for reals w, x, y, z such that $w^2 + y^2 \leq 1$.

2. n points lie in a line. How many ways are there of coloring the points with 5 colors, one of them red, so that every two adjacent points are either the same color or one or more of them is red?

3. Find all real solutions to $x^3 = 2 - y$, $y^3 = 2 - x$.

4. Show that $[1^{m/1}] + [2^{m/4}] + [3^{m/9}] + ... + [n^{m/N}] \le n + m(2^{m/4} - 1)$, where $N = n^2$, and m, n are any positive integers.

5. Find all positive integers m, n such that the roots of $x^3 - 17x^2 + mx - n^2$ are all integral.

6. A, B, C, D, E, F lie on a circle in that order. The tangents at A and D meet at P and the lines BF and CE pass through P. Show that the lines AD, BC, EF are parallel or concurrent.

7. Find positive integers m, n with the smallest possible product mn such that the number $m^m n^n$ ends in exactly 98 zeros.

8. Given an infinite sheet of squared paper. A positive integer is written in each small square. Each small square has area 1. For some $n > 2$, every two congruent polygons (even if mirror images) with area n and sides along the rulings on the paper have the same sum for the numbers inside. Show that all the numbers in the squares must be equal.

9. ABC is a triangle. K, L, M are the midpoints of the sides BC, CA, AB respectively, and D, E, F are the midpoints of the arcs BC (not containing A), CA (not containing B), AB (not containing C) respectively. Show that r + KD + LE + MF = R, where r is the inradius and R the circumradius.

22nd Austrian-Polish 1999

1. X is the set {1, 2, 3, ... , n}. How many ordered 6-tuples $(A_1, A_2, ... , A_6)$ of subsets of X are there such that every element of X belongs to 0, 3 or 6 subsets in the 6-tuple?

2. Find the best possible k, k' such that $k < v/(v + w) + w/(w + x) + x/(x + y) + y/(y + z) + z/(z + v) < k'$ for all positive reals v, w, x, y, z.

3. Given $n > 1$, find all real-valued functions $f_i(x)$ on the reals such that for all x, y we have:

$f_1(x) + f_1(y) = f_2(x) f_2(y)$

$f_2(x^2) + f_2(y^2) = f_3(x)f_3(y)$

$f_3(x^3) + f_3(y^3) = f_4(x)f_4(y)$

...

$f_n(x^n) + f_n(y^n) = f_1(x)f_1(y)$.

4. P is a point inside the triangle ABC. Show that there are unique points A_1 on the line AB and A_2 on the line CA such that P, A_1, A_2 are collinear and $PA_1 = PA_2$. Similarly, take B_1, B_2, C_1, C_2, so that P, B_1, B_2 are collinear, with B_1 on the line BC, B_2 on the line AB and $PB_1 = PB_2$, and P, C_1, C_2 are collinear, with C_1 on the line CA, C_2 on the line BC and $PC_1 = PC_2$. Find the point P such that the triangles AA_1A_2, BB_1B_2, CC_1C_2 have equal area, and show it is unique.

5. The integer sequence a_n satisfies $a_{n+1} = a_n^3 + 1999$. Show that it contains at most one square.

6. Find all non-negative real solutions to

$x_2^2 + x_1x_2 + x_1^4 = 1$
$x_3^2 + x_2x_3 + x_2^4 = 1$
$x_4^2 + x_3x_4 + x_3^4 = 1$
...
$x_{1999}^2 + x_{1998}x_{1999} + x_{1998}^4 = 1$
$x_1^2 + x_{1999}x_1 + x_1^4 = 1$.

7. Find all positive integers m, n such that $m^{n+m} = n^{n-m}$.

8. P and Q are on the same side of the line L. The feet of the perpendiculars from P, Q to L are M, N respectively. The point S is such that PS = PM and QS = QN. The perpendicular bisectors of SM and SN meet at R. The ray RS meets the circumcircle of PQR again at T. Show that S is the midpoint of RT.

9. A *valid* set is a finite set of plane lattice points and segments such that: (1) the endpoints of each segment are lattice points and it is parallel to x = 0, y = 0, y = x or y = -x; (2) two segments have at most one common point; (3) each segment has just five points in the set. Does there exist an infinite sequence of valid sets, S_1, S_2, S_3, ... such that S_{n+1} is formed by adding one segment and one lattice point to S_n?

23rd Austrian-Polish 2000

1. Find all polynomials p(x) with real coefficients such that for some n > 0, p(x+2) - p(x+3) + 2p(x+4) - 2p(x+5) + ... - n p(x+2n) + n p(x+2n+1) = 0 holds for infinitely many real x.

2. O is a vertex of a cube side 1. OABC and OADE are faces of the cube. Find the shortest distance between a point of the circle inscribed in OABC and a point of the circumcircle of OAD.

3. For $n > 2$ find all real solutions to: $x_1^3 = x_2 + x_3$, $x_2^3 = x_3 + x_4$, ... , $x_{n-2}^3 = x_{n-1} + x_n$, $x_{n-1}^3 = x_n + x_1$, $x_n^3 = x_1 + x_2$.

4. Find all positive integers n, not divisible by any primes except (possibly) 2 and 5, such that $n + 25$ is a square.

5. For which $n > 4$ can we color the vertices of a regular n-gon with 6 colors so that every 5 adjacent vertices have different colors?

6. A unit cube is glued onto each face of a central unit cube (so that the glued faces coincide). Can copies of the resulting solid fill space?

7. ABC is a triangle. The points A', B', C' lie on the lines BC, CA, AB respectively and A'B'C' is similar to ABC. Find all possible positions for the circumcenter of A'B'C'.

8. Given 27 points in the plane. Four of the points form a unit square. The other points are all inside the square. No three points are collinear. Show that we can find three points forming a triangle with area at most 1/48.

9. Show that $2 \le (1 - x^2)^2 + (1 - y^2)^2 + (1 - z^2)^2 \le (1 + x)(1 + y)(1 + z)$ for non-negative reals x, y, z with sum 1.

10. The diagram shows the plan of the castle. There are 16 nodes. Eight pairs are connected by two links each (at the four corners). How many closed paths pass through each node just once (only count a path once irrespective of whether it is traversed clockwise or counter-clockwise)? How many closed paths pass through each link once? In this case, treat paths as different if a link is traversed in opposite directions.

24th Austrian-Polish 2001

1. How many positive integers n have a non-negative power which is a sum of 2001 non-negative powers of n?

2. Take $n > 2$. Solve $x_1 + x_2 = x_3^2$, $x_2 + x_3 = x_4^2$, ... , $x_{n-2} + x_{n-1} = x_n^2$, $x_{n-1} + x_n = x_1^2$, $x_n + x_1 = x_2^2$ for the real numbers $x_1, x_2, \dots, x_n$.

3. Show that $2 < (a + b)/c + (b + c)/a + (c + a)/b - (a^3 + b^3 + c^3)/(abc) \le 3$, where a, b, c are the sides of a triangle.

4. Show that the area of a quadrilateral is at most (ac + bd)/2, where the side lengths are a, b, c, d (with a opposite c). When does equality hold?

5. Label the squares of a chessboard according to a knights' tour. So for i = 1, 2, ... , 63, the square labeled i+1 is one away from i in the direction parallel to one side of the board and two away in the perpendicular direction. Take any positive numbers $x_1, x_2, \ldots, x_{64}$ and let $y_i = 1 + x_i^2 - (x_{i-1}^2 x_{i+1})^{1/3}$ for i a white square and $1 + x_i^2 - (x_{i-1} x_{i+1}^2)^{1/3}$ for i a black square (x_0 means x_{64} and x_{65} means x_1). Show that $y_1 + y_2 + \ldots + y_{64} \ge 48$.

6. Define $a_0 = 1$, $a_{n+1} = a_n + [a_n^{1/k}]$, where k is a positive integer. Find $S_k = \{n \mid n = a_m$ for some m$\}$.

7. Show that there are infinitely many positive integers n which do not contain the digit 0, whose digit sum divides n and in which each digit that does occur occurs the same number of times. Show that there is a positive integer n which does not contain the digit 0, whose digit sum divides n, and which has k digits.

8. The top and bottom faces of a prism are regular octagons and the sides are squares. Every edge has length 1. The midpoints of the faces are M_i. A point P inside the prism is such that each ray M_iP meets the prism again in a different face at N_i. Show that $PM_1/M_1N_1 + PM_2/M_2N_2 + \ldots + PM_{10}/M_{10}M_{10} = 5$.

9. Find the largest possible number of subsets of {1, 2, ... , 2n} each with n elements such that the intersection of any three distinct subsets has at most one element.

10. A sequence of real numbers is such that the product of each pair of consecutive terms lies between -1 and 1 and the sum of every twenty consecutive terms is non-negative. What is the largest possible value for the sum of the first 2010 terms?

25th Austrian-Polish 2002

1. Find all triples (a, b, c) of non-negative integers such that $2^a + 2^b + 1$ is a multiple of $2^c - 1$.

2. Show that any convex polygon with an even number of vertices has a diagonal which is not parallel to any of its edges.

3. A line through the centroid of a tetrahedron meets its surface at X and Y, so the centroid divides the segment XY into two parts. Show that the shorter part is at least one-third of the length of the longer part.

4. Given a positive integer n, find the largest set of real numbers such that $n + (x_1^{n+1} + x_2^{n+1} + \ldots + x_n^{n+1}) \ge n\, x_1 \ldots x_n + (x_1 + \ldots + x_n)$ for all x_i in the set. When do we have equality?

5. p(x) is a polynomial with integer coefficients. Every value p(n) for n an integer is divisible by at least one of 2, 7, 11, 13. Show that every coefficient of p(x) is divisible by 2, or every coefficient is 7, or every coefficient is divisible by 11, or every coefficient is divisible by 13.

6. ABCD is a convex quadrilateral whose diagonals meet at E. The circumcenters of ABE, CDE are U, V respectively, and the orthocenters of ABE, CDE are H, K respectively. Show that E lies on the line UK iff it lies on the line VH.

7. Let N be the set of positive integers and R the set of reals. Find all functions $f : N \to R$ such that $f(x + 22) = f(x)$ and $f(x^2y) = f(x)^2f(y)$ for all x, y.

8. How many real n-tuples $(x_1, x_2, ... , x_n)$ satisfy the equations $\cos x_1= x_2$, $\cos x_2 = x_3$, ... , $\cos x_{n-1} = x_n$, $\cos x_n = x_1$?

9. A graph G has 2002 points and at least one edge. Every subgraph of 1001 points has the same number of edges. Find the smallest possible number of edges in the graph, or failing that the best lower bound you can.

10. Is it true that given any positive integer N, we can find X such that (1) any real sequence $x_0, x_1, x_2, ...$ satisfying $x_{n+1} = X - 1/x_n$ satisfies $x_k = x_{k+N}$ for all k, and (2) given a positive integer M < N, we can always find some real sequence $x_0, x_1, x_2, ...$ satisfying $x_{n+1} = X - 1/x_n$ such that $x_k = x_{k+M}$ does not hold for all k?

26th Austrian-Polish 2003

1. Find all real polynomials p(x) such that $p(x-1)p(x+1) \equiv p(x^2-1)$.

2. The sequence $a_0, a_1, a_2, ...$ is defined by $a_0 = a$, $a_{n+1} = a_n + L(a_n)$, where L(m) is the last digit of m (eg L(14) = 4). Suppose that the sequence is strictly increasing. Show that infinitely many terms must be divisible by d = 3. For what other d is this true?

3. ABC is a triangle. Take a = BC etc as usual. Take points T_1, T_2 on the side AB so that $AT_1 = T_1T_2 = T_2B$. Similarly, take points T_3, T_4 on the side BC so that $BT_3 = T_3T_4 = T_4C$, and points T_5, T_6 on the side CA so that $CT_5 = T_5T_6 = T_6A$. Show that if $a' = BT_5$, $b' = CT_1$, $c' = AT_3$, then there is a triangle A'B'C' with sides a', b', c' (a' = B'C' etc). In the same way we take points T_i' on the sides of A'B'C' and put $a'' = B'T_6'$, $b'' = C'T_2'$, $c'' = A'T_4'$.

Show that there is a triangle A"B"C" with sides a", b", c" and that it is similar to ABC. Find a"/a.

4. A positive integer m is alpine if m divides $2^{2n+1} + 1$ for some positive integer n. Show that the product of two alpine numbers is alpine.

5. A triangle with sides a, b, c has area F. The distances of its centroid from the vertices are x, y, z. Show that if $(x + y + z)^2 \le (a^2 + b^2 + c^2)/2 + 2F\sqrt{3}$, then the triangle is equilateral.

6. ABCD is a tetrahedron such that we can find a sphere k(A,B,C) through A, B, C which meets the plane BCD in the circle diameter BC, meets the plane ACD in the circle diameter AC, and meets the plane ABD in the circle diameter AB. Show that there exist spheres k(A,B,D), k(B,C,D) and k(C,A,D) with analogous properties.

7. Put $f(n) = (n^n - 1)/(n - 1)$. Show that $n!^{f(n)}$ divides $(n^n)!$. Find as many positive integers as possible for which $n!^{f(n)+1}$ does not divide $(n^n)!$.

8. Given reals $x_1 \ge x_2 \ge ... \ge x_{2003} \ge 0$, show that $x_1^n - x_2^n + x_3^n - ... - x_{2002}^n + x_{2003}^n \ge (x_1 - x_2 + x_3 - x_4 + ... - x_{2002} + x_{2003})^n$ for any positive integer n.

9. Take any 26 distinct numbers from {1, 2, ... , 100}. Show that there must be a non-empty subset of the 26 whose product is a square.

10. What is the smallest number of 5x1 tiles which must be placed on a 31x5 rectangle (each covering exactly 5 unit squares) so that no further tiles can be placed? How many different ways are there of placing the minimal number (so that further tiles are blocked)? What are the answers for a 52x5 rectangle?